Fourth order functional boundary value problems: Existence results and extremal solutions

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Dedicated to Professor Stepan Tersian on the occasion of his 60th anniversary

ABSTRACT

In this work we present two types of results for some fourth order functional boundary value problems. The first one presents an existence and location result for a problem where every boundary conditions have functional dependence. The second one states sufficient conditions for the existence of extremal solutions for functional problems with more restrict boundary functions. The arguments make use of lower and upper solutions technique, a Nagumo-type condition,an adequate version of Bolzano's theorem and existence of extremal fixed points for a suitable mapping.

Keywords: Nonlinear functional problems, lower and upper solutions, extremal solutions

1. INTRODUCTION

This paper contains two types of results for some fourth order functional boundary value problems: the first one presents an existence and location result for a problem where every boundary conditions have functional dependence on the unknown function and its first and second derivatives. The second one states sufficient conditions for the existence of extremal solutions for functional problems with more restrict boundary functions. More precisely, firstly we consider the problem composed by the functional equation

$$(1) \quad u^{(iv)}(x) = f(x, u, u', u''(x), u'''(x))$$

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with $x \in I \equiv [a, b]$, $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ a L^1- Carathéodory function and the nonlinear functional boundary conditions

$$
\begin{aligned}
L_0\left(u, u', u'', u\left(a\right)\right) &= 0, \\
L_1\left(u, u', u'', u'\left(a\right)\right) &= 0, \\
L_2\left(u, u', u'', u''\left(a\right), u'''\left(a\right)\right) &= 0, \\
L_3\left(u, u', u'', u''\left(b\right), u'''\left(b\right)\right) &= 0,
\end{aligned}
\tag{2}
$$

where L_i, $i = 0, 1, 2, 3$, are continuous functions satisfying some monotonicity assumptions to be defined later.

The second part of the work provides sufficient conditions for the existence of extremal solutions to the fourth order functional equation

$$
-\left(\phi\left(u'''(x)\right)\right)' = f(x, u''(x), u'''(x), u, u', u''),
\tag{3}
$$

for $a.a.$ $x \in]0, 1[$, with ϕ an increasing homeomorphism, $I := [0, 1]$, and $f : I \times \mathbb{R}^2 \times (C(I))^3 \to \mathbb{R}$ a L^1-Carathéodory function, coupled with the boundary conditions

$$
\begin{aligned}
0 &= L_1\left(u\left(a\right), u, u', u''\right) \tag{4} \\
0 &= L_2\left(u'\left(a\right), u, u', u''\right), \tag{5} \\
0 &= L_3\left(u''\left(a\right), u''\left(b\right), u'''\left(a\right), u'''\left(b\right), u, u', u''\right) \tag{6} \\
0 &= L_4\left(u''\left(a\right), u''\left(b\right)\right), \tag{7}
\end{aligned}
$$

where L_i, $i = 1, 2, 3, 4$, are suitable functions, with L_1 and L_2 not necessarily continuous, satisfying some monotonicity assumptions to be specified.

Due to the functional dependence in the differential equation, which nonlinearity does not need to be continuous in the independent variable and in the functional part, and in the boundary conditions covers many types of boundary value problems, such as integro-differential, with advances, delays, deviated arguments, nonlinear, Lidstone, multi-point, nonlocal, ... As example we refer the works [1, 2, 3, 12, 13, 14, 16, 17, 19, 20, 22, 25, 26, 27, 29, 30, 31, 32, 33] for nonlinear boundary conditions, and [4, 6, 7, 8, 21, 23] for functional problems. In the research for sufficient conditions to guarantee the existence of extremal solutions we refer, as example, [10, 18], for first and second order, and [5, 11], for higher orders.

The arguments used here follow standard arguments in lower and upper solutions technique, as it was suggested, for instance, in [15, 24], and for the existence of extremal solutions the followed in [11]. In short, it is considered a reduced order auxiliary problem together with two algebraic equations, the lower and upper

solutions method, a sharp version of Bolzano's theorem and the existence of extremal fixed points for a suitable operator. However the new boundary functions assumed here, (4) and (5), require other types of monotonicity in the differential equation and in the boundary conditions, and, moreover, different definitions of lower and upper solutions with their first derivatives well-ordered. Therefore, (3)-(7) can be applied to different problems, not covered by the existent literature.

2. EXISTENCE AND LOCATION RESULT

In this section presents sufficient conditions for the solvability of problem (1) – (2). The main result is an existence and location theorem, meaning that it is provided not only the existence of a solution but also its localization in an adequate strip, and for the first and second derivatives as well.

2.1 Definitions and auxiliary lemmas

A Nagumo-type growth condition, assumed on the nonlinear part, will be an important tool to set an *a priori* bound for the third derivative of the corresponding solutions.

In the following, $W^{4,1}(I)$ denotes the usual Sobolev Spaces in I, that is, the subset of $C^3(I)$ functions, whose third derivative is absolutely continuous in I and the fourth derivative belongs to $L^1(I)$.

The nonlinear part f will be a locally L^1-bounded Carathéodory function, in the following standard sense:

$f(x, \cdot, \cdot, \cdot, \cdot)$ is continuous in $(C(I))^2 \times \mathbb{R}^2$ for a.e. $x \in I$; $f(\cdot, \eta, \xi, y_0, y_1)$ is measurable for all $(\eta, \xi, y_0, y_1) \in (C(I))^2 \times \mathbb{R}^2$; and for every $R > 0$ there exists $\psi \in L^1(I)$ and a null measure set $N \subset I$ such that $|f(x, \eta, \xi, y_0, y_1)| \leq \psi(x)$ for all $(x, \eta, \xi, y_0, y_1) \in (I \setminus N) \times (C(I))^2 \times \mathbb{R}^2$ with $\|(\eta, \xi, y_0, y_1)\|_\infty \leq R$.

The functions L_i, $i = 0, 1, 2, 3$, considered in boundary conditions, must verify the following monotonicity properties:

(H_0) L_0, $L_1 : (C(I))^3 \times \mathbb{R} \to \mathbb{R}$ are continuous functions, nondecreasing in first, second and third variables;

(H_1) $L_2 : (C(I))^3 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, nondecreasing in first, second, third and fifth variables;

(H_2) $L_3 : (C(I))^3 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, nondecreasing in first, second and third variables and nonincreasing in the fifth one.

The main tool to obtain the location part is the upper and lower solutions method. However, in this case, they must be defined as a pair, which means that it is not possible to define them independently from each other. Moreover, it is pointed out that lower and upper functions, and the correspondent first derivatives, are not necessarily ordered.

To introduce "some order", it must be defined the following auxiliary functions:

For any $\alpha, \beta \in W^{4,1}(I)$ define functions $\alpha_i, \beta_i : I \to \mathbb{R}$, $i = 0, 1$, as it follows:

$$(8) \quad \alpha_1(x) = \min\{\alpha'(a), \beta'(a)\} + \int_a^x \alpha''(s)\, ds,$$

$$(9) \quad \beta_1(x) = \max\{\alpha'(a), \beta'(a)\} + \int_a^x \beta''(s)\, ds,$$

$$(10) \quad \alpha_0(x) = \min\{\alpha(a), \beta(a)\} + \int_a^x \alpha_1(s)\, ds,$$

$$(11) \quad \beta_0(x) = \max\{\alpha(a), \beta(a)\} + \int_a^x \beta_1(s)\, ds.$$

DEFINITION 2.1. *The functions $\alpha, \beta \in W^{4,1}(I)$ are a pair of lower and upper solutions for problem (1) – (2) if $\alpha'' \leq \beta''$, on I, and the following conditions are satisfied: For all $(v, w) \in A := [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$, the following inequalities hold:*

$$(12) \qquad\qquad\qquad \alpha^{(iv)}(x) \geq f(x, v, w, \alpha'', \alpha'''(x)), \, for \, a.e. x \in I,$$
$$(13) \qquad\qquad\qquad \beta^{(iv)}(x) \leq f(x, v, w, \beta'', \beta'''(x)), \, for \, a.e. x \in I,$$

$$(14) \quad
\begin{array}{rcccl}
L_0(\alpha_0, \alpha_1, \alpha'', \alpha_0(a)) & \geq & 0 & \geq & L_0(\beta_0, \beta_1, \beta'', \beta_0(a)) \\
L_1(\alpha_0, \alpha_1, \alpha'', \alpha_1(a)) & \geq & 0 & \geq & L_1(\beta_0, \beta_1, \beta'', \beta_1(a)) \\
L_2(\alpha_0, \alpha_1, \alpha'', \alpha''(a), \alpha'''(a)) & \geq & 0 & \geq & L_2(\beta_0, \beta_1, \beta'', \beta''(a), \beta'''(a)) \\
L_3(\alpha_0, \alpha_1, \alpha'', \alpha''(b), \alpha'''(b)) & \geq & 0 & \geq & L_3(\beta_0, \beta_1, \beta'', \beta''(b), \beta'''(b)).
\end{array}
$$

The Nagumo-type condition is given by next definition:

DEFINITION 2.2. *Consider $\Gamma_i, \gamma_i \in L^1(I)$, $i = 0, 1, 2$, such that $\gamma_i(x) \leq \Gamma_i(x)$, $\forall x \in I$, and the set*

$$E = \left\{ \begin{array}{l} (x, z_0, z_1, y_2, y_3) \in \, I \times (C(I))^2 \times \mathbb{R}^2 : \gamma_0(x) \leq z_0(x) \leq \Gamma_0(x), \\ \gamma_1(x) \leq z_1(x) \leq \Gamma_1(x), \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\}.$$

A function $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is said to verify a Nagumo-type condition in E if there exists $\varphi_E \in C([0, +\infty), (0, +\infty))$ such that

$$(15) \quad |f(x, y_0, y_1, y_2, y_3)| \leq \varphi_E(|y_3|),$$

for every $(x, y_0, y_1, y_2, y_3) \in E$, and

$$(16) \quad \int_r^{+\infty} \frac{t}{\varphi_E(t)} dt > \max_{x \in [a,b]} \Gamma_2(x) - \min_{x \in I} \gamma_2(x),$$

where $r \geq 0$ is given by $r := \max \left\{ \frac{\Gamma_2(b) - \gamma_2(a)}{b-a}, \frac{\Gamma_2(a) - \gamma_2(b)}{b-a} \right\}$.

Next result gives an *a priori* estimate for the third derivative of all possible solutions of (1).

LEMMA 2.3. *There exists $R > 0$ such that for every $L^1 - $Carathéodory function $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfying (15) and (16) and every solution u of (1) such that*

$$(17) \quad \gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \forall\, x \in I,$$

for $i = 0, 1, 2$, we have $\|u'''\| < R$. Moreover the constant R depends only on the functions φ and γ_i, Γ_i ($i = 0, 1, 2$) and not on the boundary conditions.

Proof. The proof is similar to [8, Lemma 2.1]. □

2.2 Existence and location theorem

In this section it is provided an existence and location theorem for the problem $(1) - (2)$. More precisely, sufficient conditions are given for, not only the existence of a solution u, but also to have information about the location of u, u', u'' and u'''.

The arguments of the proof require the following lemma, given on [28, Lemma 2]:

LEMMA 2.4. *For $z, w \in C^1(I)$ such that $z(x) \leq w(x)$, for every $x \in I$, define*

$$q(x, u) = max\{z, min\{u, w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\dfrac{d}{dx} q(x, u(x))$ *exists for a.e. $x \in I$.*

(b) *If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then*

$$\frac{d}{dx} q(x, u_m(x)) \to \frac{d}{dx} q(x, u(x)) \text{ for a.e. } x \in I.$$

Now, we can prove the the theorem:

THEOREM 2.5. *Assume that there exists a pair (α, β) of lower and upper solutions of problem (1) – (2), such that conditions (H_0), (H_1) and (H_2) hold.*
If $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, satisfying a Nagumo-type condition in

$$E_* = \left\{ \begin{array}{c} (x, z_0, z_1, y_2, y_3) \in \ I \times (C(I))^2 \times \mathbb{R}^2 : \alpha_0(x) \le z_0(x) \le \beta_0(x), \\ \alpha_1(x) \le z_1(x) \le \beta_1(x), \alpha''(x) \le y_2 \le \beta''(x) \end{array} \right\},$$

then problem (1) – (2) has at least one solution u such that

$$\alpha_0(x) \le u(x) \le \beta_0(x), \ \alpha_1(x) \le u'(x) \le \beta_1(x), \ \alpha''(x) \le u''(x) \le \beta''(x),$$

for every $x \in I$, and $|u'''(x)| \le K, \ \forall \ x \in I$, where

(18) $K = \max_{x \in I} \{R, |\alpha'''(x)|, |\beta'''(x)|\}$

and $R > 0$ is given by Lemma 2.3 referred to the set E_*.

    Proof. Define the continuous functions

(19) $\begin{aligned} \delta_i(x, y_i) &= \max\{\alpha_i(x), \min\{y_i, \beta_i(x)\}\}, \, for \, i = 0, 1, \\ \delta_2(x, y_2) &= \max\{\alpha''(x), \min\{y_2, \beta''(x)\}\} \end{aligned}$

and

$$q(z) = \max\{-K, \min\{z, K\}\} \, for all \, z \in \mathbb{R}.$$

Consider the modified problem composed by the equation

(20) $u^{(iv)}(x) = f\left(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \delta_2(x, u''(x)), q\left(\dfrac{d}{dx}(\delta_2(x, u''(x)))\right)\right)$

and the boundary conditions

(21) $\begin{aligned} u(a) &= \delta_0(a, u(a) + L_0(u, u', u'', u(a))), \\ u'(a) &= \delta_1(a, u'(a) + L_1(u, u', u'', u'(a))), \\ u''(a) &= \delta_2(a, u''(a) + L_2(u, u', u'', u''(a), u'''(a))), \\ u''(b) &= \delta_2(b, u''(b) + L_3(u, u', u'', u''(b), u'''(b))). \end{aligned}$

The proof will be proved by following several steps:

Step 1 - *Every solution u of problem (20) – (21), satisfies $\alpha''(x) \leq u''(x) \leq \beta''(x)$, $\alpha_1(x) \leq u'(x) \leq \beta_1(x)$, $\alpha_0(x) \leq u(x) \leq \beta_0(x)$ and $|u'''(x)| < K$, for every $x \in I$, with $K > 0$ given in (18).*

Let u be a solution of the modified problem (20) – (21). Assume, by contradiction, that there exists $x \in I$ such that $\alpha''(x) > u''(x)$ and let $x_0 \in I$ be such that

$$\min_{x \in I}(u - \alpha)''(x) = (u - \alpha)''(x_0) < 0.$$

As, by (21), $u''(a) \geq \alpha''(a)$ and $u''(b) \geq \alpha''(b)$, then $x_0 \in (a, b)$. So, there is $(x_1, x_2) \subset (a, b)$ such that

$$(22) \quad u''(x) < \alpha''(x), \ \forall x \in (x_1, x_2), (u - \alpha)''(x_1) = (u - \alpha)''(x_2) = 0.$$

Therefore, for all $x \in (x_1, x_2)$ it is satisfied that $\delta_2(x, u''(x)) = \alpha''(x)$ and $\frac{d}{dx}\delta_2(x, u''(x)) = \alpha'''(x)$. Now, since for all $u \in C^1(I)$ we have that $(\delta_0(\cdot, u), \delta_1(\cdot, u')) \in A$, we deduce, for *a.e.* $x \in (x_1, x_2)$,

$$
\begin{aligned}
u^{(iv)}(x) &= f\left(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right) \\
&= f(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \alpha''(x), \alpha'''(x)) \leq \alpha^{(iv)}(x).
\end{aligned}
$$

In consequence we deduce that function $(u - \alpha)'''$ is monotone nonincreasing on the interval (x_1, x_2). From the fact that $(u - \alpha)'''(x_0) = 0$, we know that $(u - \alpha)''$ is monotone nonincreasing too on (x_0, x_2), which contradicts the definitions of x_0 and x_2.

The inequality $u''(x) \leq \beta''(x)$ in I, can be proved in same way and, so,

$$(23) \quad \alpha''(x) \leq u''(x) \leq \beta''(x), \forall x \in I.$$

By (21) and (8), the following inequalities hold for every $x \in I$,

$$
\begin{aligned}
u'(x) &= u'(a) + \int_a^x u''(s)\,ds \\
&\geq \alpha_1(a) + \int_a^x \alpha''(s)\,ds = \min\{\alpha'(a), \beta'(a)\} + \int_a^x \alpha''(s)\,ds = \alpha_1(x).
\end{aligned}
$$

Analogously, it can be obtained $u'(x) \leq \beta_1(x)$, for $x \in I$.

On the other hand, by using (21), (10) and (11), the following inequalities are fulfilled:

$$u(x) \geq \alpha_0(a) + \int_a^x \alpha_1(s)\, ds = \min\{\alpha(a), \beta(a)\} + \int_a^x \alpha_1(s)\, ds = \alpha_0(x).$$

The inequality $u(x) \leq \beta_0(x)$ for every $x \in I$ is deduced in the same way.

Applying previous bounds in Lemma 2.3, for K given by (18), it is obtained the *a priori* bound $|u'''(x)| < K$, for $x \in I$. For details, see [**?**, Lemma 2.1].

Step 2 - *Problem (20) – (21) has at least one solution.*

For $\lambda \in [0,1]$ let us consider the homotopic problem given by

$$(24) \quad u^{(iv)}(x) = \lambda f\left(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right)$$

and the boundary conditions

$$(25) \quad \begin{array}{rll}
u(a) = & \lambda \delta_0(a, u(a) + L_0(u, u', u'', u(a))) & \equiv \lambda L_A, \\
u'(a) = & \lambda \delta_1(a, u'(a) + L_1(u, u', u'', u'(a))) & \equiv \lambda L_B, \\
u''(a) = & \lambda \delta_2(a, u''(a) + L_2(u, u', u'', u''(a), u'''(a))) & \equiv \lambda L_C, \\
u''(b) = & \lambda \delta_2(b, u''(b) + L_3(u, u', u'', u''(b), u'''(b))) & \equiv \lambda L_D.
\end{array}$$

Let us consider the norms in $C^3(I)$ and in $L^1(I) \times \mathbb{R}^4$, respectively,

$$\|v\|_{C^3} = \max\{\|v\|_\infty, \|v'\|_\infty, \|v''\|_\infty, \|v'''\|_\infty\}$$

and $|(h, h_1, h_2, h_3, h_4)| = \max\{\|h\|_{L^1}, \max\{|h_1|, |h_2|, |h_3|, |h_4|\}\}$.

Define the operators $\mathcal{L} : W^{4,1}(I) \subset C^3(I) \to L^1(I) \times \mathbb{R}^4$ by

$$\mathcal{L}u(x) = \left(u^{(iv)}(x), u(a), u'(a), u''(a), u''(b)\right), \quad x \in I,$$

and, for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C^3(I) \to L^1(I) \times \mathbb{R}^4$ by

$$\mathcal{N}_\lambda u(x) = \left(\begin{array}{c} \lambda f\left(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right), \\ L_A, L_B, L_C, L_D \end{array}\right)$$

Since L_0, L_1, L_2 and L_3 are continuous and f is a L^1- Carathéodory function, then, from Lemma 2.4, $\mathcal{N}_\lambda$ is continuous (see [9, Theorem 3.5] for details).

Moreover, as $\mathcal{L}^{-1}$ is compact, it can be defined the completely continuous operator $\mathcal{T}_\lambda : C^3(I) \to C^3(I)$ by $\mathcal{T}_\lambda u = \mathcal{L}^{-1}\mathcal{N}_\lambda(u)$. It is obvious that the fixed points of operator $\mathcal{T}_\lambda$ coincide with the solutions of problem (24) – (25). As $\mathcal{N}_\lambda u$ is bounded in $L^1(I) \times \mathbb{R}^4$ and uniformly bounded in $C^3(I)$, we have that any solution of the problem (24) – (25), verifies the following a priori bound $\|u\|_{C^3} \le \|\mathcal{L}^{-1}\| |\mathcal{N}_\lambda(u)| \le \bar{K}$, for some $\bar{K} > 0$ independent of λ.

In the set $\Omega = \left\{ u \in C^3(I) : \|u\|_{C^3} < \bar{K} + 1 \right\}$ the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and, by the invariance under homotopy, $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_1, \Omega, 0)$.

As the equation $x = \mathcal{T}_0(x)$ is equivalent to the problem

$$u^{(iv)}(x) = 0, \quad x \in I, \quad u(a) = u'(a) = u''(a) = u''(b) = 0,$$

which has only the trivial solution, then $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = \pm 1$. So by degree theory, the equation $x = \mathcal{T}_1(x)$ has at least one solution, that is, the problem (20) – (21) has at least one solution in Ω.

$\underline{Step\ 3}$ - Every solution u of problem (20) – (21) is a solution of (1) – (2).

Let u be a solution of the modified problem (20) – (21). By previous steps, function u fulfills equation (1). So, it will be enough to prove the following four inequalities:

(26)
$$
\begin{array}{rcl}
\alpha_0(a) \le & u(a) + L_0(u, u', u'', u(a)) & \le \beta_0(a), \\
\alpha_1(a) \le & u'(a) + L_1(u, u', u'', u'(a)) & \le \beta_1(a), \\
\alpha''(a) \le & u''(a) + L_2(u, u', u'', u''(a), u'''(a)) & \le \beta''(a), \\
\alpha''(b) \le & u''(b) + L_3(u, u', u'', u''(b), u'''(b)) & \le \beta''(b).
\end{array}
$$

Assume that

(27) $u(a) + L_0(u, u', u'', u(a)) > \beta_0(a).$

Then, by (21), $u(a) = \beta_0(a)$ and, by (H_0) and previous steps, it is obtained the following contradiction with (27):

$$u(a) + L_0(u, u', u'', u(a)) \le \beta_0(a) + L_0(\beta_0, \beta_1, \beta'', \beta_0(a)) \le \beta_0(a).$$

Applying similar arguments it can be proved that $\alpha_0(a) \le u(a) + L_0(u, u', u'', u(a))$ and $\alpha_1(a) \le u'(a) + L_1(u, u', u'', u'(a)) \le \beta_1(a)$. For the third case assume, again by contradiction, that

(28) $u''(a) + L_2(u, u', u'', u''(a), u'''(a)) > \beta''(a).$

By (21), $u''(a) = \beta''(a)$ and, as $u''(x) \leq \beta''(x)$ in I, then $u'''(a) \leq \beta'''(a)$ and, by (H_1) and (14), it is achieved this contradiction with (28):

$$
\begin{aligned}
u''(a) + L_2(u, u', u'', u''(a), u'''(a)) &\leq \beta''(a) + L_2(\beta_0, \beta_1, \beta'', \beta''(a), \beta'''(a)) \\
&\leq \beta''(a).
\end{aligned}
$$

The same technique yields the two last inequalities. □

3. EXISTENCE OF EXTREMAL SOLUTIONS

This section concerns with the presentation of sufficient conditions for the existence of extremal solutions, that is, maximal and minimal solutions, for problem (3) − (7). In short, the method considers a reduced order auxiliary problem together with two algebraic equations and applies lower and upper solutions method,a version of Bolzano's theorem and the existence of extremal fixed points for aa adeqaute operator.

3.1 Auxiliary problem

Let us consider the nonlinear second order problem

$$
\begin{aligned}
(29) \quad -(\phi(y'(x))' &= g(x, y(x), y'(x)) \quad \text{for a.a. } t \in I, \\
(30) \quad 0 &= l_1(y(a), y(b), y'(a), y'(b)), \\
(31) \quad 0 &= l_2(y(a), y(b)),
\end{aligned}
$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and $g : I \times \mathbb{R}^2 \to \mathbb{R}$ a Carathéodory function, i.e., $g(x, \cdot, \cdot)$ is a continuous function for a.a. $x \in I$, $g(\cdot, u, v)$ is measurable for all $(u, v) \in \mathbb{R}^2$, and for every $M > 0$ there exists a real-valued function $h_M \in L^1(I)$ such that for a.a. $x \in I$ and for every $(u, v) \in \mathbb{R}^2$ with $|u| \leq M$ and $|v| \leq M$ we have $|g(x, u, v)| \leq h_M(t)$.

Moreover, the function $l_1 : \mathbb{R}^4 \to \mathbb{R}$ is continuous, nondecreasing in the third variable and nonincreasing in the fourth one, and $l_2 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, nonincreasing with respect to its first variable and injective in the second argument.

We will denote by $AC(I)$ the set of absolutely continuous functions on I and by a solution of (29) we mean a function $\eta \in \mathcal{C}^1(I)$ such that $\phi(\eta') \in AC(I)$ and satisfying the differential equation almost everywhere on I.

LEMMA 3.1. *[10, Theorem 4.1] Suppose that there exist* $\alpha, \beta \in \mathcal{C}^1(I)$ *such that* $\alpha \leq \beta$ *on* I, $\phi(\alpha'), \phi(\beta') \in AC(I)$, *and*

$$-(\phi(\alpha'))'(x) \leq g(x, \alpha(x), \alpha'(x)) \quad \text{for a.a. } x \in I,$$

$$-(\phi\,(\beta'))'(x) \geq g(x, \beta(x), \beta'(x)) \quad \text{for a.a. } x \in I,$$

$$l_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b)) \geq 0 \geq l_1(\beta(a), \beta(b), \beta'(a), \beta'(b)),$$

$$l_2(\alpha(a), \alpha(b)) = 0 = l_2(\beta(a), \beta(b)).$$

Suppose that a Nagumo condition relative to α and β is satisfied, i.e., there exist functions $k \in L^p(I), \quad 1 \leq p \leq \infty$, and $\theta : [0, +\infty) \longrightarrow (0, +\infty)$ continuous, such that, for a.a. $t \in I$,

$$|g(x, u, v)| \leq k(x)\,\theta(|v|) \quad \text{for all } u \in [\alpha(t), \beta(t)] \text{ and all } v \in \mathbb{R},$$

and

$$\min\left\{ \int_{\phi(\nu)}^{+\infty} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)}\,du, \ \int_{-\infty}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)}\,du \right\} > \mu^{\frac{p-1}{p}} \|k\|_p,$$

where

$$\mu = \max_{x \in I} \beta(x) - \min_{x \in I} \alpha(x),$$

$$\nu = \frac{\max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}}{b - a},$$

$$\|k\|_p = \begin{cases} \text{ess } \sup_{x \in I} |k(x)| & , \quad p = \infty \\ \left[\int_a^b |k(x)|^p dx \right]^{\frac{1}{p}} & , \quad 1 \leq p < \infty \end{cases}$$

where "ess sup" means essential supremum and considering $(p-1)/p \equiv 1$ for $p = \infty$.

Then the problem $(29) - (31)$ has extremal solutions in

$$[\alpha, \beta] := \{\gamma \in \mathcal{C}^1(I) : \alpha \leq \gamma \leq \beta \quad \text{on } I\},$$

i.e., there exist a least and a greatest solution to the problem in the functional interval $[\alpha, \beta]$.

The Nagumo condition guarantees that the first derivative is *a priori* bounded, i.e., there exists $N > 0$, depending only on α, β, k, θ, ϕ and p, such that every solution $y \in [\alpha, \beta]$ of $(29) - (31)$ satisfies $|y'(t)| \leq N$ for all $t \in I$.

3.2 Extremal solutions

In the following, a mapping $\omega : \mathcal{C}(I) \to \mathbb{R}$ is nondecreasing if $\omega(\gamma) \leq \omega(\delta)$ whenever $\gamma(x) \leq \delta(x)$ for all $x \in I$, and ω is nonincreasing if $\omega(\gamma) \geq \omega(\delta)$ whenever $\gamma(x) \leq \delta(x)$ for all $x \in I$.

Let us consider now the initial problem (3) – (7) with the following assumptions:

(E) $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and $f : I \times \mathbb{R}^2 \times (\mathcal{C}(I))^3 \to \mathbb{R}$ satisfying:

(a) For all $(u, v, \gamma, \delta, \varepsilon) \in \mathbb{R}^2 \times (\mathcal{C}(I))^3$, $f(\cdot, u, v, \gamma, \delta, \varepsilon)$ is measurable;

(b) For a.a. $x \in I$ and all $(u, v, \gamma, \delta, \varepsilon) \in \mathbb{R}^2 \times (\mathcal{C}(I))^3$, $f(x, u, v, \cdot, \delta, \varepsilon)$, $f(x, u, v, \gamma, \cdot, \varepsilon)$ and $f(t, u, v, \gamma, \delta, \cdot)$ are nondecreasing;

(c) For a.a. $x \in I$ and all $(\gamma, \delta, \varepsilon) \in (\mathcal{C}(I))^3$, $f(x, \cdot, \cdot, \gamma, \delta, \varepsilon)$ is continuous on $\mathbb{R}^2$;

(d) For every $M > 0$ there exists a real-valued function $h_M \in L^1(I)$ such that for a.a. $x \in I$ and for every $(u, v, \gamma, \delta, \varepsilon) \in \mathbb{R}^2 \times (\mathcal{C}(I))^3$ with

$$|u| + |v| + \max_{x \in I} |\gamma(x)| + \max_{x \in I} |\delta(x)| + \max_{x \in I} |\varepsilon(x)| \leq M$$

we have $|f(x, u, v, \gamma, \delta, \varepsilon)| \leq h_M(t)$.

$(L1)$ For $i = 1, 2$, for all $\gamma, \delta, \varepsilon \in \mathcal{C}(I)$, and for all $t \in \mathbb{R}$, we have

$$\limsup_{y \to t^-} L_i(y, \gamma, \delta, \varepsilon) \leq L_i(t, \gamma, \delta, \varepsilon) \leq \liminf_{y \to t^+} L_i(y, \gamma, \delta, \varepsilon)$$

and the mappings L_i are nonincreasing in the second, third and fourth arguments.

$(L2)$ For every $\gamma, \delta, \varepsilon \in \mathcal{C}(I)$ the mappings

$$l_1 : (t, y, u, v) \in \mathbb{R}^4 \longmapsto l_1(t, y, u, v) := L_3(t, y, u, v, \gamma, \delta, \varepsilon)$$

and L_4 satisfy the conditions assumed for l_1 and l_2 in the previous section. Moveover, the operator L_3 is nondecreasing in the fifth, sixth and seventh arguments.

DEFINITION 3.2. *A function* $\alpha \in \mathcal{C}^3(I)$ *is a lower solution of (3) – (7) if* $\phi(\alpha''') \in AC(I)$ *and*

$$
\begin{aligned}
-(\phi(\alpha'''))'(x) &\le f(x, \alpha''(x), \alpha'''(x), \alpha, \alpha', \alpha'') \quad \text{for a.a. } x \in I = [a, b], \\
0 &\ge L_1(\alpha(a), \alpha, \alpha', \alpha''), \\
0 &\ge L_2(\alpha'(a), \alpha, \alpha', \alpha''), \\
0 &\le L_3(\alpha''(a), \alpha''(b), \alpha'''(a), \alpha'''(b), \alpha, \alpha', \alpha''), \\
0 &= L_4(\alpha''(a), \alpha''(b)).
\end{aligned}
$$

An upper solution is defined analogously with the reverse inequalities.

In the sequel we will use the following notation: for a couple of functions $\gamma, \delta \in \mathcal{C}(I)$ such that $\gamma \le \delta$ on I, we define

$$
[\gamma, \delta] := \{ \xi \in \mathcal{C}(I) \ : \ \gamma \le \xi \le \delta \text{ on } I \}.
$$

DEFINITION 3.3. *Let* $\alpha, \beta \in \mathcal{C}^3(I)$ *be such that* $\alpha^{(i)} \le \beta^{(i)}$ *on* I *for* $i = 0, 1, 2$. *We say that* $f : I \times \mathbb{R}^2 \times (\mathcal{C}(I))^3 \to \mathbb{R}$ *satisfies a Nagumo condition relative to* α *and* β *if there exist functions* $k \in L^p(I)$, $1 \le p \le \infty$, *and* $\theta : [0, +\infty) \longrightarrow (0, +\infty)$ *continuous, such that, for a.e.* $x \in I$, *for all* $u \in [\alpha''(t), \beta''(t)]$ *and for all* $(\gamma, \delta, \varepsilon) \in [\alpha, \beta] \times [\beta', \alpha'] \times [\alpha'', \beta'']$, *we have*

$$
|f(x, u, v, \gamma, \delta, \varepsilon)| \le k(x) \theta(|v|) \quad \text{for all } v \in \mathbb{R},
$$

and

$$
\min \left\{ \int_{\phi(\nu)}^{+\infty} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du, \int_{-\infty}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du \right\} > \mu^{\frac{p-1}{p}} \|k\|_p,
$$

where $\mu = \max_{x \in I} \beta''(x) - \min_{x \in I} \alpha''(x)$ *and*

$$
\nu = \frac{\max \{|\alpha''(a) - \beta''(b)|, |\alpha''(b) - \beta''(a)|\}}{b - a}.
$$

The following version of Bolzano's theorem plays a key role in the proof of the main result:

LEMMA 3.4. *[?, Lemma 2.3] Let* $a, b \in \mathbb{R}$, $a \le b$, *and let* $h : \mathbb{R} \to \mathbb{R}$ *be such that either* $h(a) \ge 0 \ge h(b)$ *and*

$$
\limsup_{z \to x^-} h(z) \le h(x) \le \liminf_{z \to x^+} h(z) \quad \text{for all } x \in [a, b],
$$

or $h(a) \leq 0 \leq h(b)$ and

$$\liminf_{z \to x^-} h(z) \geq h(x) \geq \limsup_{z \to x^+} h(z) \quad \text{for all } x \in [a, b].$$

Then there exist $c_1, c_2 \in [a, b]$ such that $h(c_1) = 0 = h(c_2)$ and if $h(c) = 0$ for some $c \in [a, b]$ then $c_1 \leq c \leq c_2$, i.e., c_1 and c_2 are, respectively, the least and the greatest of the zeros of h in $[a, b]$.

For the reader's convenience let us introduce some additional notation which allows more concise statements.

In $\mathcal{C}^2(I)$ we consider the standard partial ordering: Given $\gamma, \delta \in \mathcal{C}^2(I)$,

$$\gamma \widetilde{\leq} \delta \text{ if and only if } \gamma^{(i)} \leq \delta^{(i)} \text{ on } I \text{ for } i = 0, 1, 2.$$

Notice that $\mathcal{C}^2(I)$ is an ordered metric space when equipped with this partial ordering together with the usual metric, in the sense that for every $\gamma \in \mathcal{C}^2(I)$ the intervals

$$[\gamma)_{\widetilde{\leq}} = \{\delta \in \mathcal{C}^2(I) : \gamma \widetilde{\leq} \delta\} \quad \text{and} \quad (\gamma]_{\widetilde{\leq}} = \{\delta \in \mathcal{C}^2(I) : \delta \widetilde{\leq} \gamma\},$$

are closed in the corresponding topology. More details about ordered metric spaces can be seen in [18].

For $\gamma, \delta \in \mathcal{C}^2(I)$ such that $\gamma \widetilde{\leq} \delta$ define

$$[\gamma, \delta]_{\widetilde{\leq}} := \{\xi \in \mathcal{C}^2(I) : \gamma \widetilde{\leq} \xi \widetilde{\leq} \delta\}.$$

The function γ^* is the $\widetilde{\leq}$-greatest solution of $(3) - (7)$ in $[\gamma, \delta]_{\widetilde{\leq}}$ if γ^* is a solution of $(3) - (7)$ which belongs to $[\gamma, \delta]_{\widetilde{\leq}}$ and such that for any other solution $\gamma \in [\gamma, \delta]_{\widetilde{\leq}}$ we have $\gamma \leq \gamma^*$. The $\widetilde{\leq}$-least solution of $(3) - (7)$ in $[\gamma, \delta]_{\widetilde{\leq}}$ is defined analogously. If the $\widetilde{\leq}$-least and $\widetilde{\leq}$-greatest solutions of $(3) - (7)$ in $[\gamma, \delta]_{\widetilde{\leq}}$ exist we call them $\widetilde{\leq}$-extremal solutions of $(3) - (7)$ in $[\gamma, \delta]_{\widetilde{\leq}}$.

The following fixed point theorem is also useful:

LEMMA 3.5. *[18, Theorem 1.2.2] Let Y be a subset of an ordered metric space $(X, \leq)$, $[a, b]$ a nonempty order interval in Y, and $G : [a, b] \to [a, b]$ a nondecreasing mapping. If $\{Gx_n\}_n$ converges in Y whenever $\{x_n\}_n$ is a monotone sequence in $[a, b]$, then there exists x_* the least fixed point of G in $[a, b]$ and x^* is the greatest one. Moreover*

$$x_* = \min\{y \mid Gy \leq y\} \quad \text{and} \quad x^* = \max\{y \mid y \leq Gy\}.$$

The main result for problem $(3) - (7)$ is the following:

THEOREM 3.6. *Suppose that conditions* (E), $(L1)$ *and* $(L2)$ *hold, and the problem* $(3) - (7)$ *has a lower solution* α *and an upper solution* β *such that*

$$(32) \quad \alpha(a) \leq \beta(a), \quad \alpha'(a) \leq \beta'(a) \quad and \quad \alpha'' \leq \beta'' \text{ on } I.$$

If f *satisfies a Nagumo condition with respect to* α *and* β *then the problem* $(3) - (7)$ *has* $\widetilde{\leq}$-*extremal solutions in* $[\alpha, \beta]_{\widetilde{\leq}}$.

Remark that the relations (32) imply that $\alpha \leq \beta$, by successive integrations between a and $x \in]a, b]$.

Proof. For every $\gamma \in [\alpha, \beta]_{\widetilde{\leq}}$ fixed, consider the nonlinear second-order problem

$$(P_\gamma) \quad \begin{cases} -(\phi(y'))'(x) &= f(x, y(t), y'(t), \gamma, \gamma', \gamma'') \quad \text{for a.a. } t \in I, \\ 0 &= L_3(y(a), y(b), y'(a), y'(b), \gamma, \gamma', \gamma''), \\ 0 &= L_4(y(a), y(b)), \end{cases}$$

together with the two equations

$$(33) \quad 0 = L_1(w, \gamma, \gamma', \gamma''),$$
$$(34) \quad 0 = L_2(w, \gamma, \gamma', \gamma'').$$

By the assumptions, α'' and β'' are, respectively, lower and upper solutions of (P_γ), according to the definitions given in Lemma 3.1. Moreover, as the remaining conditions in Lemma 3.1 are satisfied, there exists the greatest solution of (P_γ) in $[\alpha'', \beta'']$, which will be denoted by y_γ.

According to Remark ??, there exists $N > 0$ such that

$$(35) \quad |y'_\gamma(x)| \leq N \quad \text{for all } \gamma \in [\alpha, \beta]_{\widetilde{\leq}} \text{ and all } x \in I.$$

On the other hand, we have

$$0 \geq L_1(\alpha(a), \alpha, \alpha', \alpha'') \geq L_1(\alpha(a), \gamma, \gamma', \gamma''),$$

and, similarly, $0 \leq L_1(\beta(a), \gamma, \gamma', \gamma'')$. Thus, by Lemma 3.4, the equation (33) has a greatest solution $u_a = u_a(\gamma)$ in $[\alpha(a), \beta(a)]$.

Analogously, the greatest solution of (34) in $[\alpha'(a), \beta'(a)]$ exists and it will be denoted by $u'_a = u'_a(\gamma)$.

Define, for each $x \in I$, the functional operator $G : [\alpha, \beta]_{\widetilde{\lesssim}} \to [\alpha, \beta]_{\widetilde{\lesssim}}$ by

$$G\gamma(x) := u_a + u_a'(x - a) + \int_a^x \int_a^s y_\gamma(r) dr ds.$$

In order to prove that G is nondecreasing for the ordering $\widetilde{\lesssim}$ in $[\alpha, \beta]_{\widetilde{\lesssim}}$, consider $\gamma_i \in [\alpha, \beta]_{\widetilde{\lesssim}}$ for $i = 1, 2$ such that $\gamma_1 \leq \gamma_2$. The function y_{γ_1} is a lower solution of (P_{γ_2}), and so Lemma 3.1 implies that (P_{γ_2}) has extremal solutions in $[y_{\gamma_1}, \beta'']$. In particular, the greatest solution of (P_{γ_2}) between α'' and β'' must be greater than y_{γ_1}, i.e., $y_{\gamma_2} \geq y_{\gamma_1}$ on I.

Furthermore we have

$$0 = L_1(u_a(\gamma_1), \gamma_1, \gamma_1', \gamma_1'') \geq L_1(u_a(\gamma_1), \gamma_2, \gamma_2', \gamma_2''),$$

and, as $\gamma_2 \in [\alpha, \beta]_{\widetilde{\lesssim}}$ then, by the definition of upper solution, $0 \leq L_1(\beta(a), \gamma_2, \gamma_2', \gamma_2'')$. Hence Lemma 3.4 guarantees that the equation $0 = L_1(w, \gamma_2, \gamma_2', \gamma_2'')$ has extremal solutions in $[u_a(\gamma_1), \beta(a)]$. In particular, its greatest solution between $\alpha(a)$ and $\beta(a)$ must be greater than or equal to $u_a(\gamma_1)$, i.e., $u_a(\gamma_2) \geq u_a(\gamma_1)$. In a similar way we deduce that $u_a'(\gamma_2) \geq u_a'(\gamma_1)$ and, therefore, $G\gamma_1 \leq G\gamma_2$.

Let $\{\gamma_n\}_n$ be a $\widetilde{\lesssim}$-monotone sequence in $[\alpha, \beta]_{\widetilde{\lesssim}}$. Since G is nondecreasing, the sequence $\{G\gamma_n\}_n$ is also $\widetilde{\lesssim}$-monotone and, moreover, $G\gamma_n \in [\alpha, \beta]_{\widetilde{\lesssim}}$ for all $n \in \mathbb{N}$ and $\{G\gamma_n\}_n$ is bounded in $\mathcal{C}^2(I)$.

For all $n \in \mathbb{N}$ and all $x \in I$ it can be verified that

$$(G\gamma_n)'''(x) = y_{\gamma_n}'(x),$$

and, by (35), $\{(G\gamma_n)''\}_n$ is equicontinuous on I. So, from Ascoli-Arzelá's theorem $\{(G\gamma_n)''\}_n$ is convergent in $\mathcal{C}^2(I)$. Therefore G applies $\widetilde{\lesssim}$-monotone sequences into convergent sequences and, by Lemma 3.5, G has a $\widetilde{\lesssim}$-greatest fixed point in $[\alpha, \beta]_{\widetilde{\lesssim}}$, denoted by γ^*, such that

$$(36) \quad \gamma^* = \max\{\gamma \in [\alpha, \beta]_{\widetilde{\lesssim}} : \gamma \widetilde{\lesssim} G\gamma\}.$$

As γ^* is a solution of $(3) - (7)$ in $[\alpha, \beta]_{\widetilde{\lesssim}}$, we will show that γ^* is the $\widetilde{\lesssim}$-greatest solution of $(3) - (7)$ in $[\alpha, \beta]_{\widetilde{\lesssim}}$. Let γ be an arbitrary solution of $(3) - (7)$ in $[\alpha, \beta]_{\widetilde{\lesssim}}$. Notice that the relations (4) and (5), with u replaced by γ, imply that $\gamma(a) \leq u_a(\gamma)$ and $\gamma'(a) \leq u_a'(\gamma)$. Moreover, conditions (3), (6) and (7), with u replaced by γ, imply that $\gamma'' \leq y_\gamma$. Therefore $\gamma \widetilde{\lesssim} G\gamma$ which, together with (36), yields $\gamma \widetilde{\lesssim} \gamma^*$, so γ^* is the $\widetilde{\lesssim}$-greatest solution to $(3) - (7)$ in $[\alpha, \beta]_{\widetilde{\lesssim}}$.

The existence of the $\widetilde{\lesssim}$-least solution of $(3) - (7)$ in $[\alpha, \beta]_{\widetilde{\lesssim}}$ can be proven by analogous arguments and obvious changes in the definition of the operator G. $\square$

3.3 Example

The example below does not pretend to illustrate some real phenomena, but only to show the applicability of the functional components in the equation and in the boundary conditions. Notice that, like it was referred before, this problem is not covered by the existent results.

Consider the fourth order functional differential equation

$$
(37) \quad -\frac{u^{(iv)}(x)}{1+(u'''(x))^2} \;=\; -(u''(x))^3 + |u'''(x)+1|^\xi + \max_{x\in I} u'(x) + \int_0^x u(t)dt
$$

$$
+ h(x)g\left(\max_{x\in I} u''(x)\right)
$$

where $0 \le \xi \le 2$, $I := [0,1]$, $h \in L^\infty\left(I,[0,+\infty)\right)$ and $g : \mathbb{R} \to \mathbb{R}$ a nondecreasing function, with the boundary conditions

$$
A\left(u(0)\right)^{2p+1} \;=\; -\max_{x\in I} u(x) - \sum_{j=1}^{+\infty} a_j u'\left(\xi_j\right),
$$

$$
(38) \qquad B\sqrt[3]{u'(0)} \;=\; e^{-\max_{x\in I} u''(x)},
$$

$$
Cu'''(1) \;=\; u'\left(\max\{0, x-\tau\}\right),
$$

$$
u''(0) \;=\; u''(1),
$$

where $A, B, C \in \mathbb{R}$, $0 \le \xi_j \le 1, \forall j \in \mathbb{N}$, $p \in \mathbb{N}$ and $\sum_{j=1}^{+\infty} a_j$ is a nonnegative and convergent series with sum $\overline{a}$.

This problem is a particular case of (3)-(7), where $\phi(z) = \arctan z$ (remark that $\phi(\mathbb{R}) \ne \mathbb{R}$),

$$
f\left(x,y,v,\gamma,\delta,\varepsilon\right) \;=\; -y^3 + |v+1|^\xi + \max_{x\in I} \delta(x) + \int_0^x \gamma(t)dt
$$

$$
+ \; h(x)g\left(\max_{x\in I}\varepsilon(x)\right),
$$

$$
L_1\left(t,\gamma,\delta,\varepsilon\right) \;=\; -At^{2p+1} - \max_{x\in I} \gamma(x) - \sum_{j=1}^{+\infty} a_j \delta\left(\xi_j\right),
$$

$$
L_2\left(t,\gamma,\delta,\varepsilon\right) \;=\; e^{-\max_{x\in I}\varepsilon(x)} - B\sqrt[3]{t},
$$

$$
L_3(t,y,z,v,w,\gamma,\delta,\varepsilon) \;=\; \delta\left(\max\{0, x-\tau\}\right) - Cw,
$$

$$
L_4\left(t,y\right) \;=\; y - t.
$$

The functions $\alpha(x) = -x^2 - 2x - 1$ and $\beta(x) = x^2 + 2x + 1$ are, respectively, lower and upper solutions of the problem (37)-(38) for

$$-\frac{37}{6} \le h(x)g(-2) \le h(x)g(2) \le \frac{13}{6}, \forall x \in [0,1],$$
$$A \le -3 - 3\overline{a}, B \le -e^2 \, and \, C \ge \frac{3}{2}.$$

Moreover, the homeomorphism ϕ and the nonlinearity f verify condition (E) and the Nagumo condition given by Definition 3.3 with

$$k(x) \equiv 14, \theta(v) = |v+1|^\xi, \mu = 4, \upsilon = 4.$$

The boundary functions L_i, $i = 1, 2, 3, 4$, satisfy the assumptions $(L1)$ and $(L2)$. So, by Theorem 3.6, there are $\widetilde{\le}$−extremal solutions of (37)-(38) in $[\alpha, \beta]_{\widetilde{\le}}$.

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