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Minimum-variance reduced-bias estimation of the extreme value index: A theoretical and empirical study

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In extreme value (EV) analysis, the EV index (EVI), ξ, is the primary parameter of extreme events. In this work, we consider ξ positive, that is, we assume that F is heavy tailed. Classical tail parameters estimators, such as the Hill, the Moments, or the Weissman estimators, are usually asymptotically biased. Consequently, those estimators are quite sensitive to the number of upper order statistics used in the estimation. Minimum-variance reduced-bias (RB) estimators have enabled us to remove the dominant component of asymptotic bias without increasing the asymptotic variance of the new estimators. The purpose of this paper is to study a new minimum-variance RB estimator of the EVI. Under adequate conditions, we prove their nondegenerate asymptotic behavior. Moreover, an asymptotic and empirical comparison with other minimum-variance RB estimators from the literature is also provided. Our results show that the proposed new estimator has the potential to be very useful in practice.

KEYWORDS
asymptotic bias, extreme value index, minimum asymptotic bias, semiparametric estimation, statistic of extremes

1 | INTRODUCTION

Given a sample of size n of independent and identically distributed (iid) random variables (rv's), $(X_1, \ldots, X_n)$, with a common *distribution function* (df) F, let us denote by $X_{1:n} \leq \ldots \leq X_{n:n}$ the associated ascending order statistics. Let us further assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n\}$ such that the linearly normalized maximum, that is, $(X_{n:n} - b_n)/a_n$, converges in distribution to a nondegenerate rv. Then, the limit nondegenerate distribution is necessarily the general *extreme value* (EV) distribution with df given by

$$EV_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), & 1+\xi x > 0, \qquad \text{if } \xi \neq 0, \\ \exp(-\exp(-x)), & x \in R, \qquad \text{if } \xi = 0. \end{cases} \tag{1}$$

We say then that the df F belongs to the max-domain of attraction of EV_ξ, and we consider the notation $F \in \mathcal{D}\left(EV_\xi\right)$. The shape parameter ξ, in Equation (1), is the EV index (EVI), the most important parameter of extreme events and it is related with the heaviness of the right tail. In this paper, we shall assume that F is a heavy tailed model with a Pareto right tail, that is, $\overline{F}(x) = 1 - F(x) \sim (x/c)^{-1/\xi}$, as $x \to \infty$, for some positive parameters c and ξ. These Pareto tail models have been successfully used in a variety of fields, to model events such as extreme pollution, internet traffic, intensity of magnetic storms, large insurance claims, wind speed, just to name a few. The notation $g(x) \sim h(x)$ means that $g(x)/h(x) \to 1$, as $x \to \infty$. Let $U(t) := F^{\leftarrow}(1 - 1/t)$ be the reciprocal quantile function, with $F^{\leftarrow}(t) := \inf\{x : F(x) \geq t\}$ the generalized inverse function of F. We are thus assuming the validity of the first-order condition

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\xi, \tag{2}$$

for all $x > 0$. Then, for some $c > 0$, we can write $U(t) \sim ct^\xi$, as $t \to \infty$. Semiparametric tail inference is usually based on the $k+1$ upper order statistics, $X_{n-k:n} \leq \ldots \leq X_{n:n}$. For $\xi > 0$, the classic and most popular EVI-estimator is the Hill estimator,[1]

$$\hat{\xi}^H(k) := \frac{1}{k} \sum_{i=1}^{k} i\{\ln X_{n-i+1:n} - \ln X_{n-i:n}\}, \quad 1 \leq k < n, \tag{3}$$

the average of the scaled log-spacings,

$$S_i := i\{\ln X_{n-i+1:n} - \ln X_{n-i:n}\}, \quad 1 \leq i \leq k < n. \tag{4}$$

Consistency of the Hill estimator as well of many other semiparametric estimators of ξ is achieved if $X_{n-k:n}$ is an *intermediate* order statistic,[2] that is, if

$$k = k_n \to \infty \quad \text{and} \quad k/n \to 0, \text{ as } \quad n \to \infty. \tag{5}$$

The asymptotic normality of Hill's estimator, with several types of additional assumptions, has been extensively studied by several authors (see References 3-6, among others).

Most of classic semiparametric EVI and tail related estimators have a high absolute bias for large values of k and a high variance for small values of k. The bias, due to model misspecification, makes the estimators very sensitive to the choice of k. These features led researchers in EV theory to develop reduced-bias (RB) EVI-estimators by accommodating the bias in an adequate way. We mention the first RB EVI-estimators in References 7-9 (see also References 10 and 11 and references therein). The first RB EVI-estimators, named second-order RB (SORB) EVI-estimators, were able to reduce the bias at the expense of a higher variance, the so-called bias-variance tradeoff. In order to overcome this problem, several authors considered Hall's subclass of heavy tailed models[12] characterized by

$$U(t) := ct^\xi\left(1 + dx^\rho + o(x^\rho)\right), \quad \text{as} \quad t \to \infty, \tag{6}$$

where $c > 0$, $\xi > 0$, $\rho < 0$, and $d \neq 0$, and introduced the minimum-variance RB (MVRB) or corrected-Hill (CH) EVI-estimators, dependent on an adequate estimation of second-order parameters ρ and $\beta := \rho\, d/\xi$. To reduce the bias and keep the asymptotic variance at the minimum value ξ^2, the asymptotic variance of Hill's estimator, in Equation (3), it is necessary to estimate ρ and β externally, at a level k_1 such that $k = o(k_1)$, $\hat{\rho}$ and $\hat{\beta}$ need to be consistent estimators of ρ and β, respectively, and $\hat{\rho} - \rho = o_p(1/\ln n)$. When the three parameters (ξ, ρ, β) are jointly estimated at the same level k, the asymptotic variance of the EVI-estimator turns out to be $\xi^2((1-\rho)/\rho)^4$. The simplest MVRB estimators, introduced in Reference 13, are given by

$$\hat{\xi}^{CH}(k) \equiv \hat{\xi}^{CH}_{\hat{\beta},\hat{\rho}}(k) := \hat{\xi}^H(k)\left(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right), \quad 1 \leq k < n, \tag{7}$$

and

$$\hat{\xi}^{\overline{CH}}(k) \equiv \hat{\xi}^{\overline{CH}}_{\hat{\beta},\hat{\rho}}(k) := \hat{\xi}^{H}(k) \exp\left(-\frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right), \quad 1 \leq k < n, \tag{8}$$

with $\hat{\xi}^{H}(k)$ the Hill estimator in Equation (3) and $(\hat{\beta},\ \hat{\rho})$ a pair of adequate estimators of (β, ρ). Since there is not any RB or MVRB estimator that can always dominate all the alternative estimators, many other MVRB EVI-estimators have been proposed in the literature. The simulation studies related to the CH estimator, in Equation (7), like the ones in References 14-16, show that for high values of k, the CH-estimator has usually a positive bias, overestimating thus the EVI, ξ. To solve the aforementioned systematic positive bias, a new MVRB EVI-estimator was recently introduced in the literature,[17] here denoted $\widetilde{CH}$, with the functional form

$$\hat{\xi}^{\widetilde{CH}}(k) \equiv \hat{\xi}^{\widetilde{CH}}_{\hat{\beta},\hat{\rho}}(k) := \hat{\xi}^{H}(k)\left(2 - \exp\left(\frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right)\right), \quad 1 \leq k < n. \tag{9}$$

In this work, we shall also consider two other alternative MVRB EVI-estimators introduced in Reference 18, which arise from the exponential regression model applied to the scaled log-spacings S_i in Equation (4). We shall thus consider the ML estimator

$$\hat{\xi}^{ML}(k) \equiv \hat{\xi}^{ML}_{\hat{\beta},\hat{\rho}}(k) := \hat{\xi}^{H}(k) - \hat{\beta}\left(\frac{n}{k}\right)^{\hat{\rho}}\left(\frac{1}{k}\left(\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\hat{\rho}} S_i\right)\right), \tag{10}$$

and its asymptotic second-order equivalent version

$$\hat{\xi}^{\overline{ML}}(k) \equiv \hat{\xi}^{\overline{ML}}_{\hat{\beta},\hat{\rho}}(k) := \frac{1}{k}\sum_{i=1}^{k} \exp\left(-\hat{\beta}\left(\frac{n}{i}\right)^{\hat{\rho}}\right) S_i. \tag{11}$$

Since both estimators in Equations (10) and (11) also evidence a positive bias for many heavy tailed models, in this work, we introduce a new MVRB EVI-estimator of a positive EVI, denoted by $\widetilde{ML}$, and given by

$$\hat{\xi}^{\widetilde{ML}}(k) \equiv \hat{\xi}^{\widetilde{ML}}_{\hat{\beta},\hat{\rho}}(k) := \frac{1}{k}\sum_{i=1}^{k}\left(2 - \exp\left(\hat{\beta}\left(\frac{n}{i}\right)^{\hat{\rho}}\right)\right) S_i. \tag{12}$$

The purpose of this work is to study the new MVRB estimator, in Equation (12), and to compare its efficiency with other MVRB EVI-estimators from the literature. An asymptotic comparison of some of the previous MVRB EVI-estimators can be found in References 19 and 20. Since theoretical properties of MVRB's EVI-estimators are only valid for models in Hall's class, in Equation (6), we shall also conduct numerical studies for assessing the finite sample performance of the estimators for a broader class of heavy tailed models.

The estimation of the EVI is a topic of great importance for the estimation of other important tail parameters, like extreme quantiles, also called value-at-risk (VaR) in risk management, small tail probabilities, return periods of high levels, as well as other risk measures such as the conditional tail expectation. For example, whenever dealing with heavy tailed models and for small values of p, an extreme quantile χ_p is a value such that $F(\chi_p) = 1 - p$, or equivalently,

$$\chi_p := F^{\leftarrow}(1-p) = U(1/p).$$

One can easily construct an extreme quantile estimator from the following approximation for heavy tailed models: $U(1/p) \sim cp^{-\xi}$. By replacing the scale parameter c by the corresponding estimator, proposed in Reference 12, we obtain the following estimator

$$Q^{(p)}_{\hat{\xi}}(k) \equiv \hat{\chi}_p := X_{n-k:n}\left(\frac{k}{np}\right)^{\hat{\xi}}, \tag{13}$$

where $\hat{\xi}$ is any consistent estimator of ξ computed at the level k. When $\hat{\xi}$ is replaced by the Hill estimator, $\hat{\xi}^{H}(k)$, the class of estimators in Equation (13) is named Weissman-Hill estimators,[21]

$$Q^{(p)}_{\hat{\xi}^H}(k) := X_{n-k:n}\left(\frac{k}{np}\right)^{\hat{\xi}^H}. \tag{14}$$

As noted, the estimation of extremes quantiles depends thus on the estimation of the EVI. Bias reduction of the Weissman-Hill estimator, in Equation (14), can be achieved by replacing the Hill estimator by an MVRB EVI-estimator, namely, by considering the estimator

$$Q^{(p)}_{\hat{\xi}^\bullet}(k) := X_{n-k:n}\left(\frac{k}{np}\right)^{\hat{\xi}^\bullet},$$

with $\hat{\xi}^\bullet$ denoting any arbitrary MVRB semiparametric EVI-estimator. When $\hat{\xi}^\bullet$ is replaced by the new MVRB EVI-estimator in Equation (12), we obtain the new high quantile estimator

$$Q^{(p)}_{\hat{\xi}^{\widetilde{ML}}}(k) := X_{n-k:n}\left(\frac{k}{np}\right)^{\hat{\xi}^{\widetilde{ML}}}. \tag{15}$$

The asymptotic behavior of extreme quantiles estimators is strongly related to the asymptotic behavior of the associated EVI estimators. Following References 22 and 23, we can add an extra bias-reduction factor that can lead to an improvement in the stability of the quantile estimators as functions of k. Since

$$\frac{\chi_p}{X_{n-k:n}} \overset{p}{\sim} \left(\frac{k}{np}\right)^{\xi}\left(1+\left(\left(\frac{k}{np}\right)^{\rho}-1\right)\frac{A(n/k)}{\rho}\right),$$

we can consider new quantile estimators

$$\overline{Q}^{(p)}_{\hat{\xi}^{\widetilde{ML}}}(k) := X_{n-k:n}\left(\frac{k}{np}\right)^{\hat{\xi}^{\widetilde{ML}}}\left(1+\frac{\hat{\xi}^{\widetilde{ML}}\hat{\beta}}{\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\left(\left(\frac{k}{np}\right)^{\hat{\rho}}-1\right)\right) \tag{16}$$

and

$$\overline{\overline{Q}}^{(p)}_{\hat{\xi}^{\widetilde{ML}}}(k) := X_{n-k:n}\left(\frac{k}{np}\right)^{\hat{\xi}^{\widetilde{ML}}}\exp\left(\frac{\hat{\xi}^{\widetilde{ML}}\hat{\beta}}{\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\left(\left(\frac{k}{np}\right)^{\hat{\rho}}-1\right)\right), \tag{17}$$

asymptotic equivalent, up to a second order, to the estimator $Q^{(p)}_{\hat{\xi}^{\widetilde{ML}}}(k)$, in Equation (15). As noticed in remark 1.1 of Reference 23, the asymptotic behavior of the estimators in Equations (16) and (17) remains the same if we replace $(k/np)^{\rho}$ by 0, due to the fact that $(k/np)^{\rho}\ln(k/np) = o(1)$. The simulation study of the new quantile estimators in Equations (15) to (17) is a relevant topic outside the scope of this paper and to be addressed in a future work. See chapter 4 of Reference 24 for more details on the estimation of quantiles.

The outline of the paper is as follows. In Section 2, and assuming a third-order framework, we present the asymptotic properties of the classes of EVI estimators under study, providing full information on the asymptotic bias. In Section 3, we provide some simulation results to assess the performance of the estimators under study, and finally, in Section 4, we draw some conclusions regarding the asymptotic and finite behavior of the estimators.

2 | ASYMPTOTIC PROPERTIES OF THE ESTIMATORS

2.1 | Second- and third-order conditions for a heavy right tail distribution

In order to characterize the asymptotic distributional behavior of the classic semiparametric EVI-estimators, we shall assume a second-order condition that measures the rate of convergence of in the first-order condition in Equation (2), that is, in the way $U(tx)/U(t)$ approaches x^{ξ},

$$\lim_{t\to\infty}\frac{\ln U(tx)-\ln U(t)-\xi\ln x}{A(t)}=\begin{cases}\frac{x^\rho-1}{\rho}, & \text{if } \rho<0,\\ \ln x, & \text{if } \rho=0,\end{cases} \tag{18}$$

for every $x>0$, where $\rho\le 0$ is a second-order parameter ruling the rate of convergence and $|A|$ must then be of regular variation with index ρ.[25] In this work, we shall consider $\rho<0$. To derive the asymptotic bias of the MVRB EVI-estimators considered in this article, we shall further assume a third-order condition, ruling now the rate of convergence in Equation (18), and which guarantees that,

$$\lim_{t\to\infty}\frac{\frac{\ln U(tx)-\ln U(t)-\xi\ln x}{A(t)}-\frac{x^\rho-1}{\rho}}{B(t)}=\frac{x^{\rho+\rho'}-1}{\rho+\rho'}, \tag{19}$$

for all $x>0$, where $|B|$ must then be of regular variation with index ρ'. For simplicity, it is often further assumed a slightly more restrictive third-order condition with

$$A(t)=\xi\beta t^\rho \quad \text{and} \quad B(t)=\beta'\ t^{\rho'} \quad \rho\le\rho'<0, \tag{20}$$

with $\beta\ne 0$ and $\beta'\ne 0$ the "scale" second- and third-order parameters, respectively. If $\rho=\rho'$, then $A(t)=O(B(t))$, and if $\rho<\rho'$, then $A(t)=o(B(t))$. This third-order condition holds for several models used in applications as the *Fréchet*, with df

$$F(x)=1-\exp(-x^{-1/\xi}),\quad x>0,\quad \xi>0, \tag{21}$$

the *Burr*, with df

$$F(x)=1-(1+x^{-\rho/\xi})^{1/\rho},\quad x>0,\quad \xi>0, \tag{22}$$

the *generalized pareto* (*GP*), with df

$$F(x)=1-(1+\xi x)^{-1/\xi},\quad x>0,\quad \xi>0, \tag{23}$$

the *power-pareto* with quantile function

$$F^{\leftarrow}(p)=c(1-p)^\xi p^a,\quad 0<p<1,\quad c,\xi,a>0 \tag{24}$$

and the *student*'s-*t*, with df

$$F(x)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\int_{-\infty}^{x}\left(1+\frac{z^2}{\nu}\right)^{-\frac{\nu+1}{2}}dz,\quad x\in\mathbf{R},\quad \nu>0, \tag{25}$$

where Γ denotes the (complete) gamma function. It is well known that $\xi=1/\nu$ and for $\nu=1$, Equation (25) reduces to the df of the Cauchy distribution. When $\nu\to\infty$, Equation (25) converges to the df of the standard normal distribution.

2.2 | Estimation of the second-order parameters

In order to use any MVRB EVI-estimator, it is necessary to compute adequate estimates of the second-order tail parameters ρ and β. We shall consider particular members of the class of estimators for the second-order parameter proposed in Reference 26. Such a class of estimators has been first parametrized by a tuning parameter $\tau\ge 0$, but τ can be more generally considered as a real number.[27] It is defined as follows

$$\hat\rho^{(\tau)}(k):=-\left|\frac{3\,(\mathrm{T}(k;\tau)-1)}{(\mathrm{T}(k;\tau)-3)}\right|, \tag{26}$$

where

$$\mathrm{T}(k;\tau) := \begin{cases} \dfrac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}}, & \text{if } \tau \neq 0, \\ \dfrac{\ln\left(M_n^{(1)}(k)\right) - \frac{1}{2}\ln\left(M_n^{(2)}(k)/2\right)}{\frac{1}{2}\ln\left(M_n^{(2)}(k)/2\right) - \frac{1}{3}\ln\left(M_n^{(3)}(k)/6\right)}, & \text{if } \tau = 0, \end{cases}$$

and

$$M_n^{(\alpha)}(k) := \frac{1}{k}\sum_{i=1}^{k} (\ln X_{n-i+1:n} - \ln X_{n-k:n})^{\alpha}$$

are the α-moments of the log-excesses, $\{\ln X_{n-i+1:n} - \ln X_{n-k:n}\}$, $1 \le i \le k < n$. Consistency and asymptotic normality of the estimators in Equation (26) were proved in Reference 26 and the dominant component of both the asymptotic bias and asymptotic variance of these ρ-estimators was made explicit in Reference 19. The theoretical and simulated results in References 26, 28, and 29, together with their use in RB estimation, lead us to suggest in practice the use of $\tau = 0$ for $\rho \in [-1, 0)$ and $\tau = 1$ for $\rho \in (-\infty, -1)$. Other estimators of the shape second-order parameter ρ can be found in References 28,30-34, among others.

Regarding the β-estimation, we shall next consider the β-estimators introduced in Reference 35 based on the scaled log-spacings S_i in Equation (4). On the basis of any consistent estimator $\hat{\rho}$ of the second-order parameter ρ, we shall consider the β-estimator,

$$\hat{\beta}_{\hat{\rho}}(k) = \hat{\beta}(k) = \hat{\beta}_{\hat{\rho}^{(\tau)}(k)}(k) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{\left\{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\hat{\rho}}\right)\left(\frac{1}{k}\sum_{i=1}^{k} S_i\right) - \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\hat{\rho}} S_i\right)\right\}}{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\hat{\rho}}\right)\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\hat{\rho}} S_i\right) - \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-2\hat{\rho}} S_i\right)}. \tag{27}$$

The asymptotic behavior of $\hat{\beta}_{\rho}(k)$, under the second-order framework in Equation (18), was obtained in Reference 35. The full derivation of the asymptotic behavior of $\hat{\beta}_{\hat{\rho}}(k)$ under a third-order framework was derived in Reference 19. If the second-order framework in Equation (18) holds, with $A(t) = \xi\beta t^{\rho}$, $\rho < 0$, for intermediate k-values such that $\sqrt{k}A(n/k) \to \infty$, and assuming $\hat{\rho} - \rho = o_p(1/\ln n)$, then $\hat{\beta}_{\hat{\rho}}(k)$ is consistent for the estimation of β. Moreover, we obtain

$$\hat{\beta}_{\hat{\rho}}(k) - \beta \overset{p}{\sim} -\beta\ \ln(n/k)\,(\hat{\rho}(k) - \rho)\,.$$

Regarding the estimation of the "scale" parameter β, we refer References 27 and 36. In practice, we advise the use of any intermediate level such as $k_1 = \lfloor n^{1-\epsilon} \rfloor$ for some $\epsilon > 0$, small, with $\lfloor x \rfloor$ denoting the integer part of x. The choice of ϵ is not crucial, and as a compromise between theoretical and practical considerations[19,23,37] we often take $\epsilon = 0.01$.

2.3 | Asymptotic properties of the EVI-estimators

In this section, we study the asymptotic behavior of all aforementioned EVI-estimators. First, we present without proof the following lemma.[19,38]

Lemma 1. *Under the third-order framework in Equation (19), for levels k such that Equation (5) holds, with S_i, $1 \le i \le k < n$, defined in Equation (4) and for any real $\alpha \ge 1$, the distributional representation*

$$\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{\alpha-1} S_i \overset{d}{=} \frac{\xi}{\alpha} + \frac{\xi Z_k^{(\alpha)}}{\sqrt{(2\alpha-1)k}} + \frac{A(n/k)}{\alpha-\rho} + O_p\left(\frac{A(n/k)}{\sqrt{k}}\right) + \frac{A(n/k)B(n/k)}{\alpha-\rho-\rho'}(1 + o_p(1))$$

holds, where

$$Z_k^{(\alpha)} = \sqrt{(2\alpha-1)k}\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{\alpha-1}E_i - \frac{1}{\alpha}\right), \tag{28}$$

with $\{E_i\}$ a sequence of iid standard exponential rvs, has an asymptotically standard normal distribution.

We shall now study the asymptotic behavior of the aforementioned EVI-estimators defined in Equations (3) and (7) to (12). We start by presenting the asymptotic behavior of the estimators, under a third-order framework, assuming that the second-order parameters (β, ρ) are known. Next, we shall assume that both second-order parameters are unknown and adequately estimated by Equations (26) and (27) at a high level of larger order than the level k used in the computation of the EVI.

Theorem 1. *If we assume the validity of the third-order condition, in Equation (19), with the associated functions A and B given in Equation (20) and for intermediate levels k satisfying Equation (5), we get for the rv $\hat{\xi}^{\bullet}_{\beta,\rho}(k)$ with • generally denoting $H\,(\hat{\xi}^{H}_{\beta,\rho}(k) \equiv \hat{\xi}^{H}(k))$, CH, $\overline{CH}$, $\widetilde{CH}$, ML, $\overline{ML}$ and $\widetilde{ML}$ defined in (3), (7), (8), (9), (10), (11) and (12), respectively, the asymptotic distributional representation,*

$$\hat{\xi}^{\bullet}_{\beta,\rho}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + b_1^{\bullet}A(n/k) + b_2^{\bullet}A(n/k)B(n/k) + b_3^{\bullet}A^2(n/k)(1+o_p(1)), \tag{29}$$

with $Z_k = Z_k^{(1)}$ the standard normal rv in Equation (28) with $\alpha = 1$,

$$b_1^{H} = \frac{1}{1-\rho}, \quad b_1^{CH} = b_1^{\overline{CH}} = b_1^{\widetilde{CH}} = b_1^{ML} = b_1^{\overline{ML}} = b_1^{\widetilde{ML}} = 0, \quad b_2^{CH} = b_2^{\overline{CH}} = b_2^{\widetilde{CH}} = b_2^{ML} = b_2^{\overline{ML}} = b_2^{\widetilde{ML}} = \frac{1}{1-\rho-\rho'}, \tag{30}$$

and

$$\begin{gathered} b_3^{H} = 0, \quad b_3^{CH} = -\frac{1}{\xi(1-\rho)^2}, \quad b_3^{\overline{CH}} = -\frac{1}{2\xi(1-\rho)^2}, \quad b_3^{\widetilde{CH}} = -\frac{3}{2\xi(1-\rho)^2}, \\ b_3^{ML} = -\frac{1}{\xi(1-2\rho)}, \quad b_3^{\overline{ML}} = -\frac{1}{2\xi(1-2\rho)}, \quad b_3^{\widetilde{ML}} = -\frac{3}{2\xi(1-2\rho)}. \end{gathered} \tag{31}$$

Consequently, if we work with levels k such that $\sqrt{k}A(n/k) \to \lambda$, finite,

$$\sqrt{k}\left(\hat{\xi}^{\bullet}_{\beta,\rho}(k) - \xi\right) \stackrel{d}{\to} N(\lambda b_1^{\bullet}, \xi^2), \quad \text{as } n \to \infty.$$

Moreover, if $b_1^{\bullet} = 0$, we get

$$\sqrt{k}\left(\hat{\xi}^{\bullet}_{\beta,\rho}(k) - \xi\right) \stackrel{d}{\to} N\left(\lambda_B b_2^{\bullet} + \lambda_A b_3^{\bullet}, \xi^2\right), \quad \text{as } n \to \infty,$$

provided that $\hat{\xi}^{\bullet}_{\beta,\rho}(k)$ is an MVRB EVI-estimator and k is such that $\sqrt{k}A(n/k) \to \infty$, $\sqrt{k}A^2(n/k) \to \lambda_A$ and $\sqrt{k}A(n/k)B(n/k) \to \lambda_B$, both finite.

Proof. We shall only prove the result for $\hat{\xi}^{\widetilde{ML}}$, in Equation (12). The results related with $\hat{\xi}^{H}$, $\hat{\xi}^{CH}$, $\hat{\xi}^{\overline{CH}}$, $\hat{\xi}^{\widetilde{CH}}$, $\hat{\xi}^{ML}$, and $\hat{\xi}^{\overline{ML}}$, in Equations (3), (7), (8), (9), (10), and (11), respectively, can be found in References 17,19,23, among others.

The use of the quadratic Taylor approximation of the exponential function, $e^x = 1 + x + \frac{x^2}{2} + o\left(x^2\right)$, as $x \to \infty$, enables to write the following asymptotic approximation for the $\widetilde{ML}$ estimator,

$$\begin{aligned} \hat{\xi}^{\widetilde{ML}}_{\beta,\rho}(k) &= \frac{1}{k}\sum_{i=1}^{k}\left(2 - \exp\left(\beta\left(\frac{n}{i}\right)^{\rho}\right)\right) S_i = \frac{1}{k}\sum_{i=1}^{k}\left(1 - \beta\left(\frac{n}{i}\right)^{\rho} - \frac{\beta^2}{2}\left(\frac{n}{i}\right)^{2\rho}(1+o(1))\right) S_i \\ &= \frac{1}{k}\sum_{i=1}^{k} S_i - \beta\frac{1}{k}\sum_{i=1}^{k}\left(\frac{n}{i}\right)^{\rho} S_i - \frac{\beta^2}{2}\frac{1}{k}\sum_{i=1}^{k}\left(\frac{n}{i}\right)^{2\rho} S_i(1+o_p(1)). \end{aligned}$$

Applying the result presented in Lemma 1, for $\alpha = 1$, $\alpha = 1 - \rho$, and $\alpha = 1 - 2\rho$, and with $A(n/k) = \xi\beta(n/k)^\rho$, the particular case of the result in Equation (29) follows,

$$\hat{\xi}^{\widetilde{ML}}_{\beta,\rho}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + \frac{A(n/k)B(n/k)}{1-\rho-\rho'} - \frac{3A^2(n/k)}{2\xi(1-2\rho)}(1+o_p(1)),$$

as well as the remaining of the theorem. ■

Remark 1. The asymptotic distributional representation of all the MVRB EVI-estimators under study has the same $b_1^\bullet$ and $b_2^\bullet$ components. When we compare the $b_3^\bullet$ components, in Equation (31), we get,

$$b_3^{CH} = 2b_3^{\overline{CH}} = \frac{2}{3}b_3^{\widetilde{CH}} \quad \text{and} \quad b_3^{ML} = 2b_3^{\overline{ML}} = \frac{2}{3}b_3^{\widetilde{ML}}.$$

Since the asymptotic dominant bias component of the MVRB EVI-estimators can be negative, null, or positive, no further conclusion can be drawn unless we know the true values of second- and third-order parameters.

The next theorem is stated without proof in this paper. It was proved for the CH and $\overline{CH}$ EVI-estimators in Reference 23, for the ML EVI-estimator in References 18 and 19, for $\overline{ML}$ estimators in Reference 18 and in Reference 17 for the $\widetilde{CH}$ estimator. Using similar arguments, it can be easily proved for the $\widetilde{ML}$ EVI-estimator, introduced in this article.

Theorem 2. *Under the conditions of Theorem 1 and being $(\hat{\beta}, \hat{\rho})$ consistent estimators of the second-order parameters (β, ρ), both computed on the high level k_1 of larger order than k, i.e., such that $k = o(k_1)$, and assuming that $(\hat{\rho} - \rho)\ln n = o_p(1)$,*

$$\hat{\xi}^{\bullet}_{\hat{\beta},\hat{\rho}}(k) - \hat{\xi}^{\bullet}_{\beta,\rho}(k) \stackrel{d}{=} -\frac{A(n/k)}{1-\rho}\left\{\left(\frac{\hat{\beta}-\beta}{\beta}\right) + (\hat{\rho}-\rho)\left(\ln(n/k) + \frac{1}{1-\rho}\right)\right\}(1+o_p(1)),$$

with $\bullet = \{CH, \overline{CH}, \widetilde{CH}, ML, \overline{ML}, \widetilde{ML}\}$. If we further assume that $(\hat{\beta}-\beta)/\beta \stackrel{p}{\sim} -(\hat{\rho}-\rho)\ln(n/k_1)$, a condition that holds for the estimator in Equation (27), then

$$\hat{\xi}^{\bullet}_{\hat{\beta},\hat{\rho}}(k) - \hat{\xi}^{\bullet}_{\beta,\rho}(k) \stackrel{p}{\sim} -\frac{A(n/k)}{1-\rho}(\hat{\rho}-\rho)\ln(k/k_1).$$

Consequently, $\hat{\xi}^{\bullet}_{\hat{\beta},\hat{\rho}}(k)$ is consistent for the estimation of ξ if $(\hat{\rho}-\rho)\ln(k/k_1) = o_p(1/A(n/k))$ and has an asymptotic normal distribution if $(\hat{\rho}-\rho)\ln(k/k_1) = o_p(1/(\sqrt{k}A(n/k)))$.

Remark 2. If we consider a particular class of the heavy tailed models with the functions A and B in Equation (20) with $\rho = \rho' < 0$ and being $\delta = \beta'/\beta$, $\beta \neq 0$, we have for any of the aforementioned EVI-estimators,

$$\hat{\xi}^{\bullet}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + b_1^\bullet A(n/k) + O_p\left(\frac{A(n/k)}{\sqrt{k}}\right) + b_4^\bullet A^2(n/k)(1+o_p(1)),$$

with $b_1^\bullet$ given in Equation (30) and

$$b_4^H = \frac{\delta}{\xi(1-2\rho)}, \quad b_4^{CH} = \frac{1}{\xi}\left(\frac{\delta}{1-2\rho} - \frac{1}{(1-\rho)^2}\right), \quad b_4^{\overline{CH}} = \frac{1}{\xi}\left(\frac{\delta}{1-2\rho} - \frac{1}{2(1-\rho)^2}\right),$$
$$b_4^{\widetilde{CH}} = \frac{1}{\xi}\left(\frac{\delta}{1-2\rho} - \frac{3}{2(1-\rho)^2}\right), \quad b_4^{ML} = \frac{\delta-1}{\xi(1-2\rho)}, \quad b_4^{\overline{ML}} = \frac{2\delta-1}{2\xi(1-2\rho)}, \quad b_4^{\widetilde{ML}} = \frac{2\delta-3}{2\xi(1-2\rho)}.$$

As mentioned in Reference 18, the ML estimator is expected to outperform the other MVRB EVI-estimators when the underlying model is close to Burr or to GP models, in Equations (22) and (23), respectively, that is, for heavy tailed models with $\rho = \rho'$ and when the parameter δ is close to 1.

Remark 3. Let $Bias_\infty[\hat{\xi}^\bullet(k)]$ and $Var_\infty[\hat{\xi}^\bullet(k)]$ denote, respectively, the asymptotic bias and variance of any aforementioned EVI-estimator $\hat{\xi}^\bullet(k)$. Considering the slightly more restrict class of heavy tailed models mentioned in Remark 2, the *asymptotic mean squared error* (MSE) of $\hat{\xi}^{\bullet}(k)$ is then given by

$$\text{MSE}\left[\hat{\xi}^{\bullet}(k)\right] = \text{Var}_{\infty}\left[\hat{\xi}^{\bullet}(k)\right] + \text{Bias}^2_{\infty}\left[\hat{\xi}^{\bullet}(k)\right] = \begin{cases} \frac{\xi^2}{k} + (b_1^{\bullet})^2\, A^2(n/k), & \text{if } b_1^{\bullet} \neq 0, \\ \frac{\xi^2}{k} + (b_4^{\bullet})^2\, A^4(n/k), & \text{if } b_1^{\bullet} = 0, b_4^{\bullet} \neq 0. \end{cases} \tag{32}$$

Then, the level k that minimizes the asymptotic MSE in Equation (32) is asymptotically equivalent to

$$k_0^{\bullet} := \arg\min_k \text{MSE}\left[\hat{\xi}^{\bullet}(k)\right] = \begin{cases} \left(\frac{1}{(-2\rho)(b_1^{\bullet})^2\beta^2}\right)^{\frac{1}{1-2\rho}} n^{\frac{-2\rho}{1-2\rho}}, & \text{if } b_1^{\bullet} \neq 0, \\ \left(\frac{1}{(-4\rho)(b_4^{\bullet})^2\xi^2\beta^4}\right)^{\frac{1}{1-4\rho}} n^{\frac{-4\rho}{1-4\rho}}, & \text{if } b_1^{\bullet} = 0, b_4^{\bullet} \neq 0. \end{cases} \tag{33}$$

Among the EVI-estimators under consideration in this article, only for the Hill estimator is the optimal level in Equation (33) independent of ξ.

3 | A SMALL-SCALE SIMULATION STUDY

In this section, we complement the asymptotic results in Section 2 with finite-sample properties of the EVI-estimators in Equations (3), (7), (8), (9), (10), (11), and (12). A Monte Carlo simulation study was carried out with 2000 samples of sizes $n = 100$, 150, 200, 350, 500, 1000, 1500, 2000, 3500, and 5000 from the following heavy tailed models: the Fréchet model in Equation (21) with $\xi = 0.5$, the Burr model in Equation (22), the Power-Pareto in Equation (24) with $(c, \xi, a) = (1, 0.5, 1.2)$ and the half-t with $\nu = 4$ degrees of freedom. The half-t, also known as the folded-t distribution,[39] corresponds to the absolute value of the Student's t distribution in Equation (25). We also considered the log-gamma model with df $F(x) = 1 - x^{-1/\xi}(1 + \ln(x)/\xi)$, $x > 1\ (\xi > 0)$. This distribution does not belong to Hall's class in Equation (6), but it is under the second-order condition in Equation (18), with $\rho = 0$. A more flexible two parameter log-gamma distribution and its right tail characterization can be found in Reference 40.

To assess the performance of the estimators, we obtained for each model and sample size, n, the simulated values of $\hat{\xi}_i^{\bullet}(k)$, $\bullet = \{H,\ CH, \overline{CH},\ \widetilde{CH},\ ML, \overline{ML},\ \widetilde{ML}\}$, $k = 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, 2000$, the estimates of ξ provided by the ith simulated sample. Next we obtained the Monte Carlo estimates of the mean value (E) and root MSE (RMSE),

$$\text{E}[\hat{\xi}^{\bullet}(k)] = \sum_{i=1}^{2000} \frac{\hat{\xi}_i^{\bullet}(k)}{2000}, \quad \text{RMSE}[\hat{\xi}^{\bullet}(k)] = \sqrt{\sum_{i=1}^{2000} \frac{(\hat{\xi}_i^{\bullet}(k) - \xi)^2}{2000}}, \quad 1 \le k \le n-1. \tag{34}$$

We have further computed the simulated optimum level $\hat{k}_0^{\bullet} = \arg\min_k \text{RMSE}[\hat{\xi}^{\bullet}(k))]$,

$$\text{E}[\hat{\xi}_0^{\bullet}] = \text{E}[\hat{\xi}^{\bullet}(\hat{k}_0^{\bullet})] \quad \text{and} \quad \text{RMSE}[\hat{\xi}_0^{\bullet}] = \text{RMSE}[\hat{\xi}^{\bullet}(\hat{k}_0^{\bullet})]. \tag{35}$$

In Figures 1 to 5, we present, at the left, the Monte Carlo estimates of the mean value and, at the right, the corresponding estimates of the RMSE, provided by the aforementioned EVI-estimators. The horizontal solid line, at the left plot,

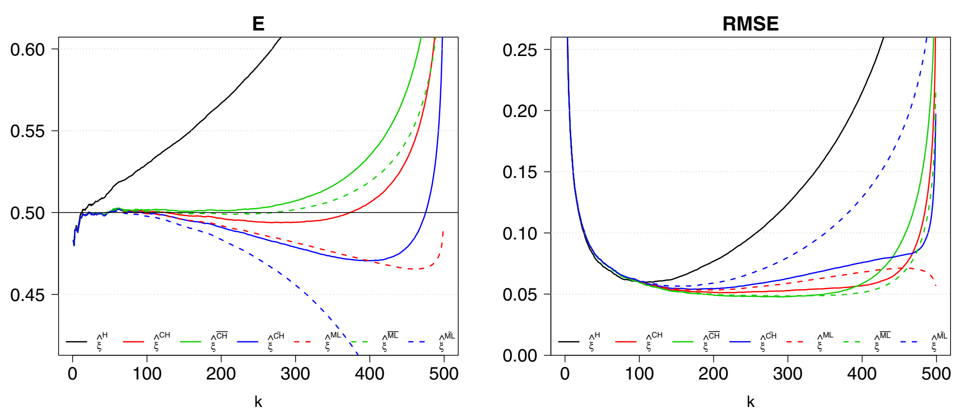

FIGURE 1 Simulated mean values (left) and RMSEs (right) of the EVI-estimators under study for samples of size $n = 500$ from a Fréchet parent with $\xi = 0.5$

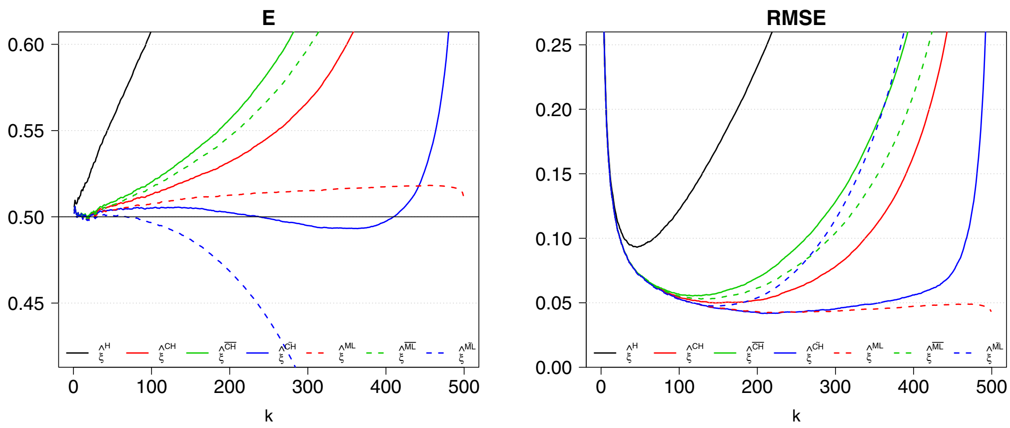

FIGURE 2 Simulated mean values (left) and RMSEs (right) of the EVI-estimators under study for samples of size $n = 500$ from a Burr parent with $\xi = 0.5$ and $\rho = -0.75$

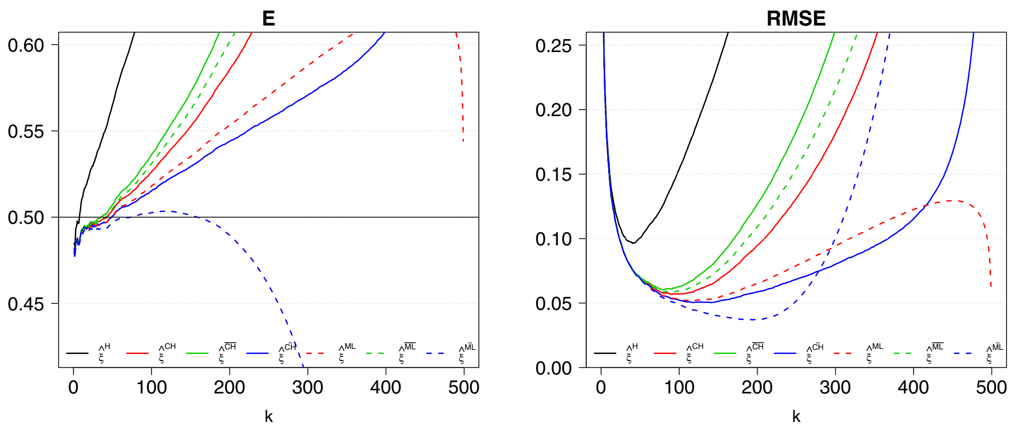

FIGURE 3 Simulated mean values (left) and RMSEs (right) of the EVI-estimators under study for samples of size $n = 500$ from a Power-Pareto parent with $(c, \xi, a) = (1, 0.5, 1.2)$

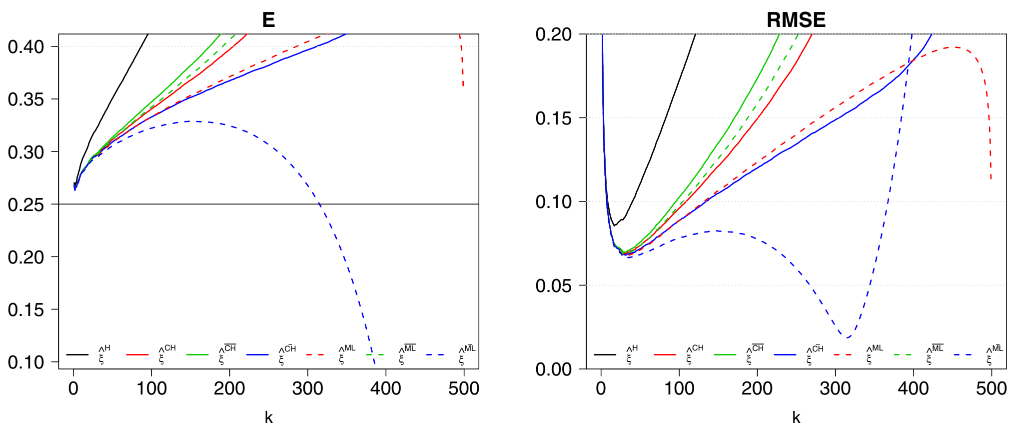

FIGURE 4 Simulated mean values (left) and RMSEs (right) of the EVI-estimators under study for samples of size $n = 500$ from a half-t parent with $\nu = 4$ $(\xi = 0.25)$

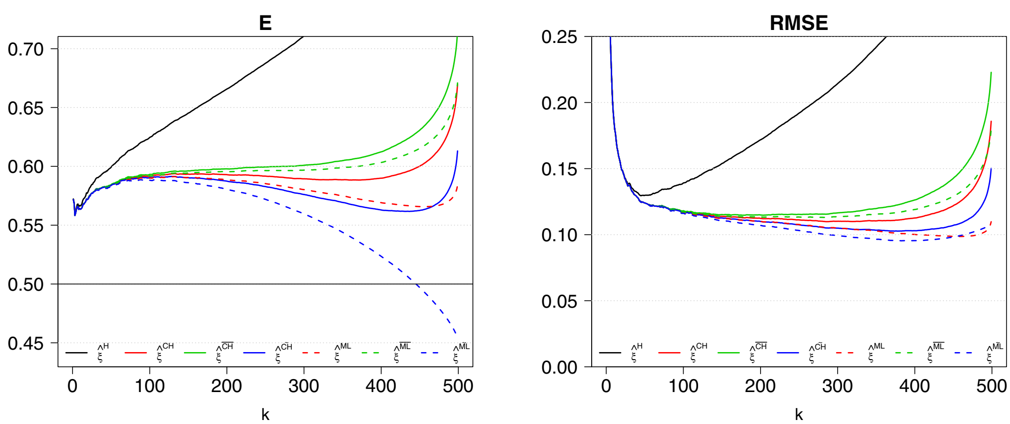

FIGURE 5 Simulated mean values (left) and RMSEs (right) of the EVI-estimators under study for samples of size $n = 500$ from a log-gamma parent with $\xi = 0.5$

indicates the true EVI value. As expected, we conclude that the Hill estimator, in Equation (3), is much more sensitive to the choice of the threshold and all MVRB EVI-estimators outperform the Hill estimator in terms of bias and RMSE. This conclusion also holds for the log-gamma model, although it is not under the considered theoretical framework. It seems that the MVRB EVI-estimators have some robustness for models outside Hall's subclass of heavy tailed distributions. For the sample size $n = 500$ considered for the plots, all MVRB EVI-estimators provide identical bias and RMSE for $k \leq 100$. For $k > 100$, some MVRB EVI-estimators perform better than other MVRB EVI-estimators: $\overline{CH}$ and $\overline{ML}$ for the Fréchet model, $\widetilde{CH}$ and ML for the Burr model and ML, $\widetilde{CH}$ and $\widetilde{ML}$ for the Power-Pareto, Half-t, and log-gamma models. This behavior depends on the underlying model and was expected (see Remark 1). Moreover, the new MVRB EVI-estimator $\widetilde{ML}$ performs well in comparison with the remaining MVRB EVI-estimators.

In Tables 1 to 3, we present the simulated values of the optimal sample fraction (the optimal level divided by the sample size), the mean value, and the RMSE of the different estimators under study. For each model, the mean value

TABLE 1 Simulated optimal sample fraction

	100	150	200	350	500	1000	1500	2000	3500	5000
	Fréchet parent with $\xi = 0.5$									
H	0.3400	0.3000	0.2750	0.2429	0.2360	0.1830	0.1520	0.1465	0.1151	0.0976
CH	0.7700	0.7667	0.6550	0.4714	0.4140	0.3090	0.3287	0.3035	0.2989	0.2928
$\overline{CH}$	0.7700	0.6600	0.6550	0.5943	0.5880	0.3430	0.3587	0.3035	0.3454	0.3412
$\widetilde{CH}$	0.4600	0.4533	0.4200	0.3800	0.3400	0.3090	0.2900	0.2975	0.2680	0.2652
ML	0.5500	0.4667	0.4500	0.4171	0.3580	0.3090	0.3000	0.3030	0.2680	0.2652
$\overline{ML}$	0.7700	0.7667	0.7550	0.7171	0.5900	0.3430	0.3580	0.3035	0.3454	0.3342
$\widetilde{ML}$	0.3700	0.3733	0.3450	0.3000	0.2880	0.2700	0.2313	0.2485	0.2391	0.2324
	Burr parent with $\xi = 0.5$ and $\rho = -0.75$									
H	0.1700	0.1467	0.1250	0.1029	0.0920	0.0740	0.0620	0.0580	0.0489	0.0452
CH	0.3900	0.3600	0.3700	0.3629	0.2940	0.2440	0.2453	0.2390	0.1929	0.1966
$\overline{CH}$	0.2900	0.2800	0.2900	0.2429	0.2340	0.2230	0.1840	0.1885	0.1637	0.1290
$\widetilde{CH}$	0.4000	0.4333	0.4450	0.4571	0.4140	0.4600	0.4853	0.4965	0.8483	0.8438
ML	0.5100	0.4333	0.5150	0.4629	0.4520	0.5650	0.5580	0.9960	0.9960	0.9992
$\overline{ML}$	0.3700	0.3533	0.2900	0.2600	0.2560	0.2440	0.2193	0.2105	0.1800	0.1662
$\widetilde{ML}$	0.3900	0.3533	0.3200	0.2886	0.2900	0.2800	0.2527	0.2470	0.2337	0.2244
	Power-Pareto parent with $(c, \xi, a) = (1, 0.5, 1.2)$									
H	0.1300	0.1200	0.1100	0.0857	0.0800	0.0730	0.0533	0.0535	0.0477	0.0362
CH	0.2700	0.2600	0.2350	0.2143	0.1820	0.1610	0.1520	0.1465	0.1403	0.1320
$\overline{CH}$	0.2200	0.2333	0.2150	0.1943	0.1600	0.1460	0.1400	0.1155	0.1151	0.0984
$\widetilde{CH}$	0.3700	0.3067	0.2700	0.2829	0.2360	0.2220	0.1827	0.1895	0.1574	0.1530
ML	0.3100	0.3067	0.2700	0.2600	0.2360	0.1830	0.1793	0.1625	0.1440	0.1420
$\overline{ML}$	0.2500	0.2400	0.2150	0.2000	0.1820	0.1600	0.1520	0.1465	0.1180	0.1116
$\widetilde{ML}$	0.3800	0.4000	0.4200	0.4000	0.3880	0.3720	0.3673	0.3550	0.3451	0.3324

(Continues)

TABLE 1 (Continued)

	100	150	200	350	500	1000	1500	2000	3500	5000
	half-t parent with $\nu = 4$ ($\xi = 0.25$)									
H	0.0800	0.0733	0.0600	0.0514	0.0340	0.0280	0.0200	0.0235	0.0169	0.0136
CH	0.1400	0.1000	0.1000	0.0771	0.0620	0.0470	0.0313	0.0245	0.0229	0.0190
$\overline{CH}$	0.1400	0.0867	0.0950	0.0743	0.0620	0.0380	0.0313	0.0245	0.0229	0.0190
$\widetilde{CH}$	0.1400	0.1133	0.1100	0.0771	0.0620	0.0470	0.0413	0.0245	0.0231	0.0190
ML	0.1400	0.1133	0.1100	0.0771	0.0620	0.0470	0.0413	0.0245	0.0231	0.0190
$\overline{ML}$	0.1400	0.1000	0.1000	0.0743	0.0620	0.0470	0.0313	0.0245	0.0229	0.0190
	0.6300	0.6267	0.6250	0.6286	0.6300	0.6310	0.6313	0.6300	0.6291	0.6286
	Log-gamma parent with $\xi = 0.5$									
H	0.2500	0.1933	0.1750	0.1257	0.0900	0.0900	0.0673	0.0440	0.0277	0.0272
CH	0.7000	0.6733	0.6450	0.5743	0.5720	0.6420	0.0853	0.0675	0.0474	0.0272
$\overline{CH}$	0.6500	0.6533	0.5650	0.5543	0.3540	0.0920	0.0853	0.0675	0.0474	0.0272
$\widetilde{CH}$	0.5400	0.6733	0.6450	0.6486	0.7480	0.7980	0.8187	0.8435	0.8551	0.8628
ML	0.7000	0.1000	0.0700	0.0629	0.9160	0.0560	0.9167	0.9175	0.9160	0.9186
$\overline{ML}$	0.7000	0.6733	0.6450	0.5743	0.5720	0.1180	0.0853	0.0675	0.0474	0.0272
	0.4400	0.5400	0.5350	0.6486	0.7660	0.8850	0.9060	0.9275	0.9263	0.9310

closest to the target value ξ and the smallest RMSE are written in **bold**. Note that the direct estimation of the optimal level of the MVRB EVI-estimators has limited interest for practitioners, since it depends not only on the value of the EVI itself, but also on the value of second- and third-order parameters. We observe that the optimal sample fraction is much larger for the MVRB EVI-estimators than for the Hill estimator. The MVRB EVI-estimators have also a smaller absolute bias and RMSE than the Hill estimator. Again, the best EVI-estimator in terms of bias or RMSE depends on the underlying model.

4 | CONCLUSIONS

In this article, we have compared the classical Hill estimator of the EVI with several MVRB EVI-estimators from the literature. A new MVRB EVI-estimator has also been introduced in this article. We have established the asymptotic limiting distribution of the EVI-estimators under a third-order framework and we have illustrated their performance with a Monte Carlo simulation study. Overall, we have concluded that MVRB EVI-estimators are much less sensitive to the choice of the threshold and are never less efficient than the Hill estimator, for the same value of k. In terms of simulated bias and RMSE, the proposed estimator $\widetilde{ML}$ is better than previous MVRB EVI-estimators, for several heavy tailed models. From a practical point of view, this new MVRB EVI-estimator is an important addition to the existing literature, since for central values of k, the reduction of the bias depends on the combination of EVI-estimator and tail model. Moreover, data-adaptive procedures for choosing both the most adequate MVRB EVI-estimator and the corresponding optimal level k is a relevant topic that needs to be addressed in future current work. We believe that such a research work would enhance the importance of the MVRB EVI-estimators in practical applications.

TABLE 2 Simulated mean values, at optimal levels

	100	150	200	350	500	1000	1500	2000	3500	5000
	Fréchet parent with $\xi = 0.5$									
H	0.5563	0.5468	0.5414	0.5365	0.5358	0.5263	0.5213	0.5213	0.5153	0.5132
CH	0.4898	**0.4935**	0.4896	**0.4939**	0.4959	**0.5000**	**0.4997**	**0.5008**	**0.4997**	**0.4997**
$\overline{CH}$	0.5208	0.5065	**0.5079**	0.5065	0.5064	0.5025	0.5026	0.5029	0.5016	0.5012
$\widetilde{CH}$	0.4724	0.4790	0.4837	0.4913	0.4942	0.4975	0.4984	0.4988	0.4987	0.4987
ML	0.4690	0.4799	0.4834	0.4897	0.4939	0.4978	0.4983	0.4989	0.4990	0.4990
$\overline{ML}$	**0.5055**	0.5078	0.5090	0.5080	**0.5015**	0.5012	0.5012	0.5020	0.5005	0.5003
$\widetilde{ML}$	0.4684	0.4755	0.4809	0.4898	0.4925	0.4964	0.4985	0.4987	0.4980	0.4984
	Burr parent with $\xi = 0.5$ and $\rho = -0.75$									
H	0.5987	0.5831	0.5708	0.5592	0.5526	0.5446	0.5376	0.5364	0.5328	0.5299
CH	0.5432	0.5327	0.5327	0.5282	0.5212	0.5150	0.5155	0.5149	0.5124	0.5125
$\overline{CH}$	0.5446	0.5384	0.5384	0.5266	0.5251	0.5224	0.5177	0.5182	0.5164	0.5125
$\widetilde{CH}$	**0.5167**	**0.5075**	**0.5060**	**0.5012**	**0.5019**	**0.4982**	**0.4973**	**0.4968**	**0.5010**	**0.4999**
ML	0.5250	0.5173	0.5182	0.5137	0.5132	0.5129	0.5130	0.5122	0.5126	0.5104
$\overline{ML}$	0.5503	0.5423	0.5317	0.5252	0.5241	0.5215	0.5194	0.5181	0.5160	0.5146
$\widetilde{ML}$	0.4794	0.4815	0.4868	0.4886	0.4880	0.4884	0.4917	0.4924	0.4944	0.4954
	Power-Pareto parent with $(c, \xi, a) = (1, 0.5, 1.2)$									
H	0.5907	0.5796	0.5731	0.5526	0.5494	0.5463	0.5338	0.5343	0.5287	0.5219
CH	0.5504	0.5447	0.5359	0.5289	0.5220	0.5166	0.5140	0.5134	0.5111	0.5098
$\overline{CH}$	0.5482	0.5505	0.5419	0.5337	0.5233	0.5195	0.5171	0.5114	0.5099	0.5065
$\widetilde{CH}$	0.5461	0.5347	0.5261	0.5266	0.5206	0.5176	0.5111	0.5126	0.5073	0.5066
ML	0.5395	0.5382	0.5296	0.5276	0.5243	0.5145	0.5136	0.5117	0.5078	0.5077
$\overline{ML}$	0.5482	0.5438	0.5350	0.5298	0.5259	0.5199	0.5174	0.5167	0.5091	0.5080
$\widetilde{ML}$	**0.4935**	**0.4919**	**0.4843**	**0.4890**	**0.4916**	**0.4945**	**0.4946**	**0.4963**	**0.4968**	**0.4979**
	half-t parent with $\nu = 4$ $(\xi = 0.25)$									
H	0.3449	0.3364	0.3266	0.3184	0.3017	0.2958	0.2882	0.2904	0.2831	0.2800
CH	0.3290	0.3150	0.3131	0.3038	0.2979	0.2918	0.2841	0.2796	0.2784	0.2767
$\overline{CH}$	0.3327	0.3123	0.3136	0.3044	0.2989	0.2882	0.2844	0.2798	0.2786	0.2769
$\widetilde{CH}$	0.3249	0.3162	0.3132	0.3023	0.2969	0.2911	0.2881	0.2793	0.2784	0.2765
ML	0.3225	0.3145	0.3120	0.3014	0.2962	0.2907	0.2878	0.2791	0.2782	0.2764
$\overline{ML}$	0.3278	0.3139	0.3125	0.3026	0.2975	0.2915	0.2839	0.2794	0.2783	0.2766
$\widetilde{ML}$	**0.2530**	**0.2539**	**0.2515**	**0.2499**	**0.2498**	**0.2504**	**0.2498**	**0.2498**	**0.2500**	**0.2500**
	Log-gamma parent with $\xi = 0.5$									
H	0.6324	0.6230	0.6172	0.6076	0.5961	0.5963	0.5910	0.5840	0.5757	0.5759
CH	0.5519	0.5686	0.5736	0.5847	0.5897	0.5937	0.5868	0.5852	0.5809	0.5740
$\overline{CH}$	0.5769	0.5883	0.5889	0.5960	0.5971	0.5857	0.5869	0.5852	0.5809	0.5740
$\widetilde{CH}$	0.5319	0.5432	0.5521	0.5657	0.5657	0.5699	0.5714	0.5727	0.5734	0.5739
ML	0.5347	0.5757	0.5741	0.5761	0.5657	0.5809	0.5706	0.5720	0.5723	0.5727
$\overline{ML}$	0.5712	0.5826	0.5856	0.5924	0.5963	0.5891	0.5865	0.5850	0.5808	0.5740
$\widetilde{ML}$	**0.5260**	**0.5382**	**0.5453**	**0.5444**	**0.5309**	**0.5079**	**0.5037**	**0.4989**	**0.5002**	**0.4994**

TABLE 3 Simulated RMSE at optimal levels

	100	150	200	350	500	1000	1500	2000	3500	5000
	Fréchet parent with $\xi = 0.5$									
H	0.1051	0.0902	0.0818	0.0674	0.0597	0.0457	0.0395	0.0365	0.0296	0.0262
CH	0.0885	0.0769	0.0703	0.0585	0.0512	0.0387	0.0333	0.0289	0.0227	0.0190
$\overline{CH}$	0.0805	0.0693	**0.0631**	**0.0536**	**0.0477**	**0.0377**	**0.0323**	**0.0283**	**0.0222**	**0.0185**
$\widetilde{CH}$	0.1019	0.0864	0.0782	0.0626	0.0539	0.0399	0.0343	0.0298	0.0233	0.0195
ML	0.0984	0.0844	0.0766	0.0615	0.0532	0.0395	0.0340	0.0295	0.0231	0.0194
$\overline{ML}$	**0.0797**	**0.0689**	0.0635	0.0539	0.0484	0.0380	0.0325	0.0285	0.0223	0.0186
$\widetilde{ML}$	0.1095	0.0923	0.0834	0.0657	0.0564	0.0411	0.0354	0.0305	0.0240	0.0200
	Burr parent with $\xi = 0.5$ and $\rho = -0.75$									
H	0.1675	0.1445	0.1291	0.1055	0.0931	0.0739	0.0644	0.0600	0.0509	0.0453
CH	0.1051	0.0857	0.0764	0.0576	0.0498	0.0358	0.0305	0.0274	0.0233	0.0206
$\overline{CH}$	0.1125	0.0927	0.0825	0.0627	0.0553	0.0411	0.0352	0.0322	0.0269	0.0237
$\widetilde{CH}$	0.0990	0.0800	0.0698	0.0491	**0.0417**	**0.0264**	**0.0214**	**0.0176**	**0.0115**	**0.0097**
ML	**0.0960**	**0.0785**	**0.0687**	**0.0489**	0.0423	0.0279	0.0235	0.0205	0.0175	0.0159
$\overline{ML}$	0.1072	0.0886	0.0790	0.0602	0.0529	0.0389	0.0337	0.0307	0.0257	0.0228
$\widetilde{ML}$	0.1026	0.0844	0.0748	0.0549	0.0474	0.0330	0.0270	0.0238	0.0186	0.0159
	Power-Pareto parent with $(c, \xi, a) = (1, 0.5, 1.2)$									
H	0.1784	0.1522	0.1376	0.1092	0.0963	0.0769	0.0672	0.0600	0.0487	0.0430
CH	0.1113	0.0926	0.0820	0.0650	0.0567	0.0415	0.0346	0.0312	0.0245	0.0210
$\overline{CH}$	0.1183	0.0999	0.0883	0.0705	0.0604	0.0447	0.0376	0.0338	0.0262	0.0225
$\widetilde{CH}$	0.1004	0.0837	0.0739	0.0579	0.0501	0.0365	0.0312	0.0277	0.0218	0.0185
ML	0.0992	0.0838	0.0742	0.0592	0.0516	0.0382	0.0324	0.0289	0.0227	0.0194
$\overline{ML}$	0.1122	0.0946	0.0839	0.0671	0.0584	0.0431	0.0363	0.0329	0.0255	0.0219
$\widetilde{ML}$	**0.0832**	**0.0671**	**0.0583**	**0.0438**	**0.0370**	**0.0254**	**0.0211**	**0.0185**	**0.0143**	**0.0120**
	half-t parent with $\nu = 4$ ($\xi = 0.25$)									
H	0.1468	0.1273	0.1159	0.0975	0.0854	0.0694	0.0620	0.0560	0.0476	0.0439
CH	0.1129	0.0986	0.0903	0.0764	0.0688	0.0576	0.0515	0.0469	0.0402	0.0376
$\overline{CH}$	0.1160	0.1002	0.0921	0.0775	0.0696	0.0581	0.0518	0.0471	0.0404	0.0377
$\widetilde{CH}$	0.1095	0.0965	0.0882	0.0752	0.0679	0.0571	0.0511	0.0467	0.0401	0.0375
ML	0.1070	0.0946	0.0869	0.0743	0.0673	0.0567	0.0509	0.0465	0.0399	0.0374
$\overline{ML}$	0.1114	0.0974	0.0895	0.0758	0.0684	0.0573	0.0513	0.0468	0.0401	0.0375
$\widetilde{ML}$	**0.0626**	**0.0378**	**0.0365**	**0.0225**	**0.0186**	**0.0128**	**0.0107**	**0.0093**	**0.0070**	**0.0059**
	Log-gamma parent with $\xi = 0.5$									
H	0.1820	0.1680	0.1556	0.1400	0.1294	0.1141	0.1079	0.1035	0.0950	0.0897
CH	0.1285	**0.1200**	**0.1171**	0.1111	0.1098	0.1056	0.1026	0.0996	0.0932	0.0885
$\overline{CH}$	0.1280	0.1243	0.1217	0.1163	0.1147	0.1080	0.1027	0.0996	0.0932	0.0885
$\widetilde{CH}$	0.1448	0.1239	0.1174	0.1069	0.1027	0.0926	0.0884	0.0862	0.0816	0.0795
ML	0.1292	0.1838	0.1823	0.1495	0.0986	0.1146	0.0843	0.0821	0.0782	0.0767
$\overline{ML}$	**0.1258**	0.1217	0.1196	0.1141	0.1130	0.1077	0.1025	0.0995	0.0931	0.0885
$\widetilde{ML}$	0.1513	0.1277	0.1192	**0.1040**	**0.0954**	**0.0732**	**0.0618**	**0.0527**	**0.0399**	**0.0326**

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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