**ON THE NATURE OF NATURAL NUMBERS**

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Young man, in mathematics you don't understand things. You just get used to them.

(J. Von Neumann)

We learn how to count at a very young age. So much so that the numbers 0, 1, 2, 3, ... become intuitive and natural[[1]](#footnote-1). However, numbers are indeed very sophisticated abstractions. For example, the word ‘one’ can be used to refer to a cat, or a table, or a person, or indeed to any “one” thing. But the “oneness” is not connected to any particular object, it is more of an abstraction about every possible “one” object.

Someone may ask: *what do you mean exactly by the word ‘one’? How would you define it?*. I have asked these very questions to many different people and got surprising, creative and interesting answers which were nevertheless usually “wrong” or, at least, insufficient as a definition. Some of the most interesting “wrong” answers are reproduced below. The first one was “I am the number one”, which seems to be a quite self-centred answer but reminded me of the roman numeral “I” for one. A second interesting answer was “the number ‘1’ is your priority when thinking of/about anything in the universe”. A third interesting answer was “It's the number related to your finger, you can count 1, 2, 3, 4 fingers”, which is not a definition of the number but is related to the fact that we use base 10, i.e., digits from 0 to 9, mostly because we have 10 fingers. Looking the word up in a dictionary may feel disappointing as well. Many times, it is defined using words like “unit” or “single thing” which are just different ways of saying “one” and provide little or no explanation nor meaning for the word that is supposed to be defined. Alternatively, some people say that it is the smallest non-zero natural number. However, this just replaces the original question by a different set of questions: *What is a natural number? What does it mean to be non-zero? What is the number zero?*. It seems that natural languages are rather inadequate to address this sort of questions. However, it is possible to use tools from set theory to give a satisfactory definition not only of the number “one” but of all “natural” numbers. I will sketch those definitions below.

Since I will only be dealing with natural numbers, unless otherwise said, the word “number” always means “natural number”.

Also, I will consider the set of numbers to be the set ℕ:={0, 1, 2, 3, ...}. The reason to include zero in the set of natural numbers will be justified below.

**A very brief historical tour**

Giving credit to Von Neumann's quote above we don't really understand natural numbers, we just got used to them. In fact, there are still some apparently simple problems concerning natural numbers for which no one knows the answer. For example, the so called Goldbach’s conjecture which states that every even number greater than 2 can be written as the sum of two prime numbers[[2]](#footnote-2).

Nevertheless, numbers seem to be very “natural” objects. In our daily life we use them with relative ease, as if they were simple objects which are easy to understand. In all fairness we had a lot of time to get used to them. Natural numbers are present in the history of mankind for a very long time, since mankind first had the need to count[[3]](#footnote-3). A clue for the fact that numbers are indeed sophisticated abstractions is that

*Various present-day populations in Oceania, America, Asia and Africa whose languages contain only the words for one, two and many, but who nonetheless understand one-for-one parities perfectly well, use notches on bones or wooden sticks to keep a tally.* [Ifrah]

In my opinion there are mainly two things to retain from the previous quote. The first one is that some of the so-called “primitive” peoples have or have had difficulties with numbers above two or have no need to use such “large numbers”, mainly due to their abstract nature. The second thing is that the notion of a one-to-one correspondence is useful to count these “large numbers”[[4]](#footnote-4) .

The word “calculus” that gave origin to the word “calculate” derives from the Latin word meaning stone. There is a story saying that before going to war soldiers would leave a stone in a pile and remove it when they got back. In this way, the number of remaining stones, corresponded to the number of soldiers lost in the battle. This was a one-for-one way to count the number of casualties.

The Ancient Greeks introduced the notion of prime number[[5]](#footnote-5). The Fundamental Theorem of Arithmetic states that every integer greater than 1 is either prime or the product of primes, and that this product is unique, up to the order of the factors. Due to this result, in order to understand the natural numbers, it is in some sense enough to understand the primes. However, even though the definition of prime number is quite simple to understand and the Greeks where able to show some properties about prime numbers, to this day primes remain somewhat mysterious.

Reflections about what are the numbers and what can we know about them occupied the mind of great mathematicians, logicians and philosophers. Among them I would like to highlight the works of Frege, Dedekind, Peano, Cantor, Gödel and Von Neumann (these and many other interesting works can be consulted in [van Heijenoort]).

The natural numbers were, and in some sense still are, a source for philosophic discussion. Pertinent questions still don't (and maybe never will) have satisfactory answers. For example: *Are numbers discovered or invented? Are mathematical objects, and in particular natural numbers, “real”?*. For Plato and the Platonists, the mathematical objects exist indeed in an ideal world, a "third realm" different from both the external world, i.e., the ``real'' world and from the internal world of consciousness. Other philosophical views such as nominalism and formalism deny this assertion[[6]](#footnote-6). According to Dedekind

*In view of this freeing of the elements from any other content (abstraction)*

*one is justified in calling the numbers a free creation of*

*the human mind* [Dedekind].

Kronecker saw the natural numbers as an exception among the mathematical objects and famously said that

*God made the natural numbers. Everything else is the work of man.*

Many questions arise from these ideas. I would like to encourage the reader to think about the following ones: *Are natural numbers special entities among the mathematical objects? If so, in what sense? What is the nature of numbers and their role in life and in mathematics? In particular, are numbers created or discovered by man?*

**Is zero natural?**

Considering the number zero to be natural is not consensual. There are advantages and disadvantages to consider 0 a natural number. Let us start with some of the disadvantages. First of all, for many people, including many mathematicians, the number 0 is not “natural” at all[[7]](#footnote-7). People usually start counting 1, 2, 3, ... and not 0, 1, 2, .... In fact, it was with much reluctance that the concept of zero was introduced in mathematics. For example, the Ancient Greeks did not have the number 0 and the Babylonians had a sort of “weak zero” that was not considered a proper number but merely a device to facilitate the writing of some large numbers and to carry out some calculations[[8]](#footnote-8). In fact, the number zero as we know it today was only accepted by the European mathematicians around the 13th century and for the general people in Europe only around the 15th century[[9]](#footnote-9). Also, in a sense one can see the number 0 as the inverse of infinity and the latter is by no means considered to be a number. Consider the sequence defined by $u\_{n}=\frac{1}{n}$. This sequence is very important in mathematics and is usually called the *harmonic sequence*. If zero is not a natural number, then the harmonic sequence is defined for all natural numbers. Otherwise, we have to say that it is defined for all natural numbers except zero, which is admittedly not so nice. Of course, this a somewhat artificial argument because one could ask *doesn’t the same problem happen with the sequence* $v\_{n}$ *defined by* $v\_{n}=\frac{1}{\left(n-1\right)}$*?*

There are however some big advantages for considering 0 to be natural. I will present a few of them:

* We currently live in the era of computers and smartphones. These machines use a binary system, only using 0's and 1's and start counting by 0.
* When doing integer division by $n$, there are $n$ possible rests: from 0 to $n-1$.
* The degree of a polynomial can be zero, as can be the order of a derivative.
* But, to me the most important reason is that, as we shall see in the next section, it allows for a nice definition of natural number.

**A formal definition**

In Zermelo-Fraenkel set theory (usually abbreviated **ZF** or **ZFC** if one includes the so-called Axiom of Choice) it is possible to build most of the mathematics used by working mathematicians using only the symbol ‘$\in $’ and some axioms[[10]](#footnote-10). The first one, which is the basis for the whole construction is the existence of a set which contains no elements. This is called the *empty set* and it is represented by the symbol $∅$.

Formally, the natural numbers are introduced via the axioms of **ZFC**. In fact, the existence of the set of natural numbers is guaranteed by the so-called axiom of infinity. This axiom postulates the existence of infinite sets by stating that there exists at least one inductive set **I**.

Natural numbers can be constructed in different ways. Here we will use Von Neumann's ordinals.

We define $0≔∅$ and if $n \ne 0$ then we define $n+1≔n ∪\{n\}$.

A set S is an *ordinal* if and only if S is strictly well-ordered with respect to set membership and every element of S is also a subset of S. The definition of natural number above means that a natural number $n$ is equal to the set of all its predecessors, i.e., $n=\{0,1,…,n-1\}$. In this way, natural numbers are indeed finite ordinals. It may seem strange at first, but it is possible to write $4 \in 5$

 because $5=\{0,1,2,3,4\}$. Also, this construction provides a natural way to define a well-order over the natural numbers. Given two natural numbers $n,m$ we say that $m$ is less than $n$ and write $m<n$ if and only if $m \in n$.

**References**

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1. I would agree that the number 0 is usually not included in the first experiences with counting. Nevertheless, children will often have the notion of “none” or “nothing” (sometimes expressed just by the word “no”) which is in my opinion another way to say “zero”. Evidence for this is, for example, the word “không” in the Vietnamese language which may be translated into English as “zero” or as “not” and is also used to say “none” or “nothing”. [↑](#footnote-ref-1)
2. [↑](#footnote-ref-2)
3. For a detailed history of numbers and deep reflections on the origins and development of numbers the reader can consult [Ifrah]. [↑](#footnote-ref-3)
4. Cantor would later use this one-for-one notion to compare the size of different infinite sets and to establish that there are as many natural numbers as there are rationals. [↑](#footnote-ref-4)
5. A prime number is a number which is only divisible by 1 and itself. For example, 2, 3, 5 and 7 are prime but 9 is not, because 9=3×3. [↑](#footnote-ref-5)
6. The reader interested in these, and others, philosophical currents and their impact in mathematics may consult for example [Shapiro]. [↑](#footnote-ref-6)
7. Isn’t it sort of “natural” to have zero million euros, though? [↑](#footnote-ref-7)
8. Much like what we do in our current number system, for example to distinguish the number 102 from the number 12, but with barely any other function. [↑](#footnote-ref-8)
9. For more on the history and importance of the number zero, I highly recommend the book [Seife]. [↑](#footnote-ref-9)
10. This is clearly not the place for a full exposition about **ZFC** but the interested reader may consult for example [HrbacekJech]. [↑](#footnote-ref-10)