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ON THE LOWER AND UPPER SOLUTION METHOD FOR HIGHER ORDER FUNCTIONAL BOUNDARY VALUE PROBLEMS

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The authors consider the n^{th}-order differential equation

$$-\big(\phi(u^{(n-1)}(x))\big)' = f(x, u(x), \ldots, u^{(n-1)}(x)),$$

for $x \in (0,1)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $n \geq 2$, $I := [0,1]$, and $f : I \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, together with the boundary conditions

$$\begin{aligned}
&g_i\big(u, u', \ldots, u^{(n-2)}, u^{(i)}(1)\big) = 0, \quad i = 0, \ldots, n-3,\\
&g_{n-2}\big(u, u', \ldots, u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\big) = 0,\\
&g_{n-1}\big(u, u', \ldots, u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\big) = 0,
\end{aligned}$$

where $g_i : (C(I))^{n-1} \times \mathbb{R} \to \mathbb{R}$, $i = 0, \ldots, n-3$, and g_{n-2}, g_{n-1} : $(C(I))^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying certain monotonicity assumptions.
The main result establishes sufficient conditions for the existence of solutions and some location sets for the solution and its derivatives up to order $(n-1)$. Moreover, it is shown how the monotone properties of the nonlinearity and the boundary functions depend on n and upon the relation between lower and upper solutions and their derivatives.

1. INTRODUCTION

In this paper, we consider the n^{th}-order boundary value problem consisting of the differential equation

$$-\big(\phi(u^{(n-1)}(x))\big)' = f(x, u(x), \ldots, u^{(n-1)}(x)), \tag{1}$$

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for $x \in (0,1)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $n \geq 2$, $I := [0,1]$, and $f : I \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, together with the boundary conditions

$$\begin{aligned} &g_i\big(u, u', \ldots, u^{(n-2)}, u^{(i)}(1)\big) = 0, \quad i = 0, \ldots, n-3, \\ &g_{n-2}\big(u, u', \ldots, u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\big) = 0, \\ &g_{n-1}\big(u, u', \ldots, u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\big) = 0, \end{aligned} \tag{2}$$

where $g_i : (C(I))^{n-1} \times \mathbb{R} \to \mathbb{R}$, $i = 0, \ldots, n-3$, and $g_{n-2}, g_{n-1} : (C(I))^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying certain monotonicity assumptions that will be described below.

This type of boundary value problem includes a wide range of equations, problems and applications that are improved by this work. As examples, we refer the reader to the papers [**6, 9, 13, 14**] for higher order separated problems, to [**4, 5, 8, 10, 11, 12, 16, 18**], for multipoint cases, and to [**1, 2, 3, 15**], for higher order functional problems.

The method used here is suggested by that in [**7**]. However, since the first $(n-3)$ boundary conditions include values at the right end point of the interval, the general result requires some features that were not evident in previous works. So the contributions of this paper emanate not just from the main theorem itself and its applications, but also from the consequences that can be drawn from it. In this sense, we wish to point out the following features.

- For $n \geq 3$, the order relation between the lower and upper solutions and their derivatives up to and including the order $(n-3)$ is not relevant. In fact, these orders depend whether n is odd or even and on the relation between the $(n-2)$-nd derivatives of the lower and upper solutions. Moreover, the sets yielding the location of the derivatives $u^{(i)}$, $i = 0, \ldots, n-2$, of the solution u of the problem (1)–(2), are defined by the derivatives of the lower and upper solutions being well ordered or in reverse order (see Remark 1).

- The assumptions on the monotonic behavior of the functions in the boundary data depend on the parity of n (see assumptions (H_1) and (H_2)).

The arguments follow the standard lower and upper solutions technique together with a Nagumo-type condition, to control the growth of $u^{(n-1)}$, and a fixed-point result. We should also point out that due to the truncation technique that we use, we do not need to require the usual assumption that $\phi(\mathbb{R}) = \mathbb{R}$.

2. PRELIMINARY RESULTS AND DEFINITIONS

In this section, we will provide some definitions and results to be used later in the paper.

Let L^p, $1 \le p \le \infty$, be the usual spaces with the standard norms

$$||u||_p = \begin{cases} \left(\int_0^1 |u(t)|^p \mathrm{d}t\right)^{1/p}, & 1 \le p < \infty, \\ \sup\{|u(t)| : t \in I\}, & p = \infty. \end{cases}$$

A function $f : I \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *Carathéodory function* if it satisfies the following conditions:

(i) For each $y \in \mathbb{R}^n$ the function $f(\cdot, y)$ is measurable on I;

(ii) For a. e. $x \in I$ the function $f(x, \cdot)$ is continuous on $\mathbb{R}^n$;

(iii) For each compact set $K \subset \mathbb{R}^n$ there is a function $\psi_K \in L^1(I)$ such that $|f(x, y)| \le \psi_K(x)$ for a. e. $x \in I$ and all $y \in K$.

The following Nagumo-type condition will play an important role in obtaining an *a priori* estimate for the derivative $u^{(n-1)}$.

Definition 1. *Given a subset $E \subset I \times \mathbb{R}^n$, a function $f : I \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo-type condition in the set*

$$E := \{(x, y_0, \dots, y_{n-1}) \in I \times \mathbb{R}^n : m_j(x) \le y_j \le M_j(x),\ j = 0, \dots, n-2\},$$

with $m_j, M_j \in C(I, \mathbb{R})$ such that

$$m_j(x) \le M_j(x) \quad \text{for all} \quad x \in I \quad \text{and} \quad j = 0, \dots, n-2,$$

if there is $h_E \in C(\mathbb{R}_0^+, (0, +\infty))$ such that

$$|f(x, y_0, \dots, y_{n-1})| \le h_E(|y_{n-1}|),\ \forall (x, y_0, \dots, y_{n-1}) \in E, \tag{3}$$

and

$$\int_{\phi(r)}^{\phi(\infty)} \frac{|\phi^{-1}(s)|}{h_E(|\phi^{-1}(s)|)} \, \mathrm{d}s > \max_{x \in [0,1]} M_{n-2}(x) - \min_{x \in [0,1]} m_{n-2}(x), \tag{4}$$

for $r \ge 0$, where

$$r := \max\{M_{n-2}(1) - m_{n-2}(0), M_{n-2}(0) - m_{n-2}(1)\}. \tag{5}$$

Our first two lemmas are taken from [**6**].

Lemma 1. ([**6**, Lemma 2]) *Let m_j, $M_j \in C([0,1], \mathbb{R})$ with*

$$m_j(x) \le M_j(x) \quad \text{for all} \quad x \in I \quad \text{and} \quad j = 0, \dots, n-2,$$

and let $f : E \to \mathbb{R}$ be a Carathéodory function satisfying a Nagumo-type condition in E. Then there exists $R > 0$ (depending only on m_{n-2}, M_{n-2}, and h_E) such that every solution $u(x)$ of (1) with

$$m_j(x) \le u^{(j)}(x) \le M_j(x) \quad \text{for all} \quad x \in I \quad \text{and} \quad j = 0, \dots, n-2, \tag{6}$$

satisfies

$$\left\| u^{(n-1)} \right\|_\infty < R.$$

Our next lemma guarantees the existence and uniqueness of solutions to a problem related to (1)–(2).

Lemma 2. ([**6**, Lemma 3]) *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be an increasing homeomorphism such that* $\varphi(0) = 0$ *and* $\varphi(\mathbb{R}) = \mathbb{R}$, *let* $p : [0,1] \to \mathbb{R}$ *with* $p \in L^1([0,1])$, *and let* A_i, B, $C \in \mathbb{R}$, $i = 0, \dots, n-3$. *Then the problem*

$$\left\{ \begin{array}{c} -\left(\varphi\left(u^{(n-1)}(x)\right)\right)' = p(x), \quad \text{for a. e. } x \in [0,1], \\ u^{(i)}(1) = A_i, \quad i = 0, \dots, n-3, \\ u^{(n-2)}(0) = B, \\ u^{(n-2)}(1) = C, \end{array} \right. \tag{7}$$

has a unique solution given by

$$u(x) = B + \int_0^x \varphi^{-1}\left(\tau_v - \int_0^s p(r)\mathrm{d}r\right)\mathrm{d}s$$

if $n = 2$, *and*

$$u(x) = \sum_{k=0}^{n-3} (-1)^k A_k \frac{(1-x)^k}{k!} + (-1)^n \int_x^1 \frac{(s-x)^{n-3}}{(n-3)!} v(s)\mathrm{d}s$$

if $n \geq 3$, *where*

$$v(x) := B + \int_0^x \varphi^{-1}\left(\tau_v - \int_0^s p(r)\mathrm{d}r\right)\mathrm{d}s$$

and $\tau_v \in \mathbb{R}$ *is the unique solution of the equation*

$$C - B = \int_0^1 \varphi^{-1}\left(\tau_v - \int_0^s p(r)\mathrm{d}r\right)\mathrm{d}s. \tag{8}$$

Some properties of truncated functions that we will need later are given in the next lemma.

Lemma 3. ([**17**, Lemma 2]) *Let* z, $w \in C(I)$ *with* $z(x) \leq w(x)$ *and for every* $x \in I$, *define*

$$q(x,u) = \max\{z, \min\{u, w\}\}.$$

Then, for each $u \in C^1(I)$, *the following properties hold*:

(*a*) $\dfrac{\mathrm{d}}{\mathrm{d}x} q(x, u(x))$ *exists for a.e.* $x \in I$;

(*b*) *If* $u, u_m \in C^1(I)$ *and* $u_m \to u$ *in* $C^1(I)$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}x} q(x, u_m(x)) \to \frac{\mathrm{d}}{\mathrm{d}x} q(x, u(x)) \quad \text{for a.e. } x \in I.$$

In the sequel, we will assume that the continuous functions $g_i : (C(I))^{n-1} \times \mathbb{R} \to \mathbb{R}$, $i = 0, \ldots, n-3$, and g_{n-2}, $g_{n-1} : (C(I))^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ have different behavior depending on whether n is even or odd. More precisely we have the following.

(i) For n even, we say that the boundary functions satisfy assumption (H_1) if the following conditions hold:

- $g_j(y_0, y_1, \ldots, y_{n-1})$ are nondecreasing in $y_0, y_2, \ldots, y_{n-2}$, and nonincreasing in $y_1, y_3, \ldots, y_{n-3}$, for j even and $0 \le j \le n-4$;
- $g_k(y_0, y_1, \ldots, y_{n-1})$ are nonincreasing in $y_0, y_2, \ldots, y_{n-2}$, and nondecreasing in $y_1, y_3, \ldots, y_{n-3}$ for k odd and $1 \le k \le n-3$;
- $g_{n-2}(y_0, y_1, \ldots, y_{n-1}, y_n)$ is nondecreasing in $y_0, y_2, \ldots, y_{n-2}$ and y_n, and nonincreasing in $y_1, y_3, \ldots, y_{n-3}$;
- $g_{n-1}(y_0, y_1, \ldots, y_{n-1}, y_n)$ is nondecreasing in $y_0, y_2, \ldots, y_{n-2}$ and nonincreasing in $y_1, y_3, \ldots, y_{n-3}$ and y_n.

(ii) For n odd, we say that the boundary functions satisfy (H_2) if the following conditions hold:

- $g_j(y_0, y_1, \ldots, y_{n-1})$ are nondecreasing in $y_0, y_2, \ldots, y_{n-3}$ and nonincreasing in $y_1, y_3, \ldots, y_{n-2}$ for j even and $0 \le j \le n-3$;
- $g_k(y_0, y_1, \ldots, y_{n-1})$ are nonincreasing in $y_0, y_2, \ldots, y_{n-3}$ and nondecreasing in $y_1, y_3, \ldots, y_{n-2}$ for k odd and $1 \le k \le n-4$;
- $g_{n-2}(y_0, y_1, \ldots, y_{n-1}, y_n)$ is nonincreasing in $y_0, y_2, \ldots, y_{n-3}$ and nondecreasing in $y_1, y_3, \ldots, y_{n-2}$ and y_n;
- $g_{n-1}(y_0, y_1, \ldots, y_{n-1}, y_n)$ is nonincreasing in $y_0, y_2, \ldots, y_{n-3}$ and y_n and nondecreasing in $y_1, y_3, \ldots, y_{n-2}$.

We let $AC(I)$ denote the set of absolutely continuous function on I. These functions will be used as lower and upper solutions as defined as follows.

Definition 2. *Let $n \ge 2$. A function $\alpha \in C^{n-1}(I)$ with $\phi\big(\alpha^{(n-1)}(x)\big) \in AC(I)$ is a lower solution of the problem* (1)–(2) *if*

$$-\big(\phi(\alpha^{(n-1)}(x))\big)' \le f(x, \alpha(x), \alpha'(x), \ldots, \alpha^{(n-1)}(x)), \tag{9}$$

for $x \in (0,1)$ a.e., and

(i) *for n even,*

$$\begin{aligned} &g_j\big(\alpha, \alpha', \ldots, \alpha^{(n-2)}, \alpha^{(j)}(1)\big) \ge 0, \text{ for } j \text{ even and } 0 \le j \le n-4, \\ &g_k\big(\alpha, \alpha', \ldots, \alpha^{(n-2)}, \alpha^{(k)}(1)\big) \le 0, \text{ for } k \text{ odd and } 1 \le k \le n-3, \end{aligned} \tag{10}$$

(ii) *for n odd,*

$$
\begin{aligned}
&g_j\big(\alpha,\alpha',\ldots,\alpha^{(n-2)},\alpha^{(j)}(1)\big)\le 0,\ \text{for } j \text{ even and } 0\le j\le n-3,\\
&g_k\big(\alpha,\alpha',\ldots,\alpha^{(n-2)},\alpha^{(k)}(1)\big)\ge 0,\ \text{for } k \text{ odd and } 1\le k\le n-4,
\end{aligned}
\tag{11}
$$

and

(iii) *in both cases,*

$$
\begin{aligned}
g_{n-2}\big(\alpha,\alpha',\ldots,\alpha^{(n-2)},\alpha^{(n-2)}(0),\alpha^{(n-1)}(0)\big)\ge 0,\\
g_{n-1}\big(\alpha,\alpha',\ldots,\alpha^{(n-2)},\alpha^{(n-2)}(1),\alpha^{(n-1)}(1)\big)\ge 0.
\end{aligned}
$$

Similarly, a function $\beta\in C^{n-1}(I)$ with $\phi\big(\beta^{(n-1)}(x)\big)\in AC(I)$ is an upper solution of the problem (1)–(2) *if the reverse inequalities hold in each case.*

3. EXISTENCE AND LOCATION THEOREM

Our main result, Theorem 1 below, is an existence and location theorem, as is usual in using the lower and upper solution technique. However, in this case, the strips are bounded by well ordered (or reverse ordered) lower and upper solutions and their corresponding derivatives. Therefore, for a more clear notation, we define the following functions:

$$
\gamma_i(x)=\min_{x\in I}\big\{\alpha^{(i)}(x),\beta^{(i)}(x)\big\}\quad\text{and}\quad \Gamma_i(x)=\max_{x\in I}\big\{\alpha^{(i)}(x),\beta^{(i)}(x)\big\},
\tag{12}
$$

for each $i=0,\ldots,n-2$.

Theorem 1. *Let $f: I\times\mathbb{R}^n\to\mathbb{R}$ be a L^1-Carathéodory function. Assume that α and β are lower and upper solutions of problem* (1)–(2), *respectively, such that*

$$
\alpha^{(n-2)}(x)\le\beta^{(n-2)}(x)\quad \text{for all}\quad x\in I,
\tag{13}
$$

$$
(-1)^m\,\alpha^{(n-2-m)}(1)\le(-1)^m\,\beta^{(n-2-m)}(1),\ m=1,\ldots,n-2,
\tag{14}
$$

f satisfies the Nagumo-type condition (3) *in the set*

$$
E_*=\big\{(x,y_0,\ldots,y_{n-1})\in I\times\mathbb{R}^n:\gamma_i(x)\le y_i\le\Gamma_i(x),\ i=0,\ldots,n-2\big\},
$$

and

$$
\begin{aligned}
f(x,\alpha(x),\ldots,\alpha^{(n-3)}(x),y_{n-2},y_{n-1})&\le f(x,y_0,\ldots,y_{n-1})\\
&\le f(x,\beta(x),\ldots,\beta^{(n-3)}(x),y_{n-2},y_{n-1}),
\end{aligned}
\tag{15}
$$

for fixed x, y_{n-2}, y_{n-1}, and $\gamma_k(x)\le y_k\le\Gamma_k(x)$, $k=0,\ldots,n-3$, for all $x\in I$. Moreover, if n is even and the boundary functions satisfy (H_1), or n is odd and the *boundary functions satisfy* (H_2), *then the problem* (1)–(2) *has at least one solution* u *such that*

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x),$$

for $i = 0, \ldots, n-2$, *and*

$$-R \leq u^{(n-1)}(x) \leq R,$$

for every $x \in I$, *with*

$$R > \max\{\beta^{(n-2)}(1) - \alpha^{(n-2)}(0), \beta^{(n-2)}(0) - \alpha^{(n-2)}(1), \|\alpha^{(n-1)}\|_\infty, \|\beta^{(n-1)}\|_\infty\}. \tag{16}$$

Remark 1. Integrating (13) in $[x, 1]$ and applying (14), causes the derivatives of the lower and upper solutions to change order. That is, for every $x \in I$,

$$\begin{aligned}
\alpha^{(n-3)}(x) &\geq \beta^{(n-3)}(x),\\
\alpha^{(n-4)}(x) &\leq \beta^{(n-4)}(x),\\
&\vdots\\
\alpha(x) &\leq \beta(x),
\end{aligned}$$

if n is even. For n odd, the iteration will end with $\alpha(x) \geq \beta(x)$ in I.

Since the relation between the lower and upper solutions depends on n, and their derivatives can be well ordered or in reversed order, this issue does not have the same relevance for $n \geq 3$ as it does for first and second order problems. As a consequence, the same can be said for the variation of the nonlinear function f as can be seen in (15).

Proof of Theorem 1. For $i = 0, \ldots, n-2$, consider the continuous truncations

$$\delta_i\,(x, w) = \begin{cases} \Gamma_i(x), & w > \Gamma_i(x),\\ w, & \gamma_i(x) \leq w \leq \Gamma_i(x),\\ \gamma_i(x), & w < \gamma_i(x), \end{cases} \tag{17}$$

where $\gamma_i(x)$ and $\Gamma_i(x)$ are given by (12). For R given by (16), consider the functions

$$\xi(y) = \max\{-R, \min\{y,\ R\}\} \tag{18}$$

and $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(y) = \begin{cases} \phi(y), & \text{if } |y| \leq R,\\ \dfrac{\phi(R) - \phi(-R)}{2R}\,y + \dfrac{\phi(R) + \phi(-R)}{2}, & \text{if } |y| > R. \end{cases}$$

Define the modified problem composed of the differential equation

$$\begin{aligned}
&-\left(\varphi\left(u^{(n-1)}(x)\right)\right)'\\
&= f\left(x, \delta_0(x, u), \ldots, \delta_{n-2}\left(x, u^{(n-2)}\right), \xi\left(\frac{\mathrm{d}}{\mathrm{d}x}\delta_{n-2}\left(x, u^{(n-2)}\right)\right)\right) \equiv F_u(x)
\end{aligned} \tag{19}$$

and the boundary conditions

$$
\begin{aligned}
&u^{(i)}(1) = \delta_i\big(1, u^{(i)}(1) + g_i(u, \dots, u^{(n-2)}, u^{(i)}(1))\big), \quad i = 0, \dots, n-3, \\
&u^{(n-2)}(0) = \delta_{n-2}\big(0, u^{(n-2)}(0) + g_{n-2}(u, \dots, u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0))\big), \\
&u^{(n-2)}(1) = \delta_{n-2}\big(1, u^{(n-2)}(1) + g_{n-1}(u, \dots, u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1))\big).
\end{aligned} \tag{20}
$$

A function $u \in C^{n-1}(I)$ such that $\varphi \circ u^{(n-1)} \in AC(I)$ is a solution of problem (19)–(20) if it satisfies the above equalities.

Step 1: *Every solution of problem* (19)–(20) *satisfies*

$$\gamma_i(x) \le u^{(i)}(x) \le \Gamma_i(x), \quad \textit{for} \quad i = 0, \dots, n-2, \tag{21}$$

$$-R \le u^{(n-1)}(x) \le R \tag{22}$$

in I.

Let u be a solution of (19)–(20). For $i = n-2$, we have $\gamma_{n-2}(x) = \alpha^{(n-2)}(x)$ and $\Gamma_{n-2}(x) = \beta^{(n-2)}(x)$. Assume, for the sake of a contradiction, that the second inequality in (21) does not hold and define

$$\max_{x \in [0,1]} (u - \beta)^{(n-2)}(x) := (u - \beta)^{(n-2)}(x_0) > 0.$$

By (20), $u^{(n-2)}(0) \le \beta^{(n-2)}(0)$ and $u^{(n-2)}(1) \le \beta^{(n-2)}(1)$. So there is an $x_0 \in (0,1)$ with $u^{(n-1)}(x_0) = \beta^{(n-1)}(x_0)$ and there is $\varepsilon > 0$ such that

$$u^{(n-2)}(x_0 + \varepsilon) = \beta^{(n-2)}(x_0 + \varepsilon)$$

and $u^{(n-2)}(x) > \beta^{(n-2)}(x)$ on $[x_0, x_0 + \varepsilon)$.

On $(x_0, x_0 + \varepsilon)$, by Definition 2, (15), (17), (18) and (16), we have

$$
\begin{aligned}
-\big(\varphi(u^{(n-1)}(x))\big)' &= f\left(x, \delta_0\left(x, u\right), \dots, \delta_{n-2}(x, u^{(n-2)}), \xi\left(\frac{\mathrm{d}}{\mathrm{d}x}\delta_{n-2}(x, u^{(n-2)})\right)\right) \\
&= f\big(x, \delta_0\left(x, u\right), \dots, \delta_{n-3}(x, u^{(n-3)}), \beta^{(n-2)}(x), \beta^{(n-1)}(x)\big) \\
&\le f\big(x, \beta(x), \dots, \beta^{(n-3)}, \beta^{(n-2)}\left(x\right), \beta^{(n-1)}(x)\big) \\
&\le -\big(\phi(\beta^{(n-1)}(x))\big)' = -\big(\varphi(\beta^{(n-1)}(x))\big)'.
\end{aligned}
$$

Therefore, $u^{(n-1)}(x) \ge \beta^{(n-1)}(x)$ on $(x_0, x_0 + \varepsilon)$, which contradicts the definition of the interval $[x_0, x_0 + \varepsilon)$. Hence, $u^{(n-2)}(x) \le \beta^{(n-2)}(x)$ for every $x \in I$. By an analogous argument, it can be shown that $\alpha^{(n-2)}(x) \le u^{(n-2)}(x)$ in I.

Integrating the inequalities

$$\alpha^{(n-2)}(x) \le u^{(n-2)}(x) \le \beta^{(n-2)}(x),$$

in $[x, 1]$ and applying (14) and (20), we obtain

$$\alpha^{(n-3)}(x) \ge u^{(n-3)}(x) \ge \beta^{(n-3)}(x).$$

Repeated integrations show that for n even,

$$\alpha^{(j)}(x) \leq u^{(j)}(x) \leq \beta^{(j)}(x) \text{ for } j \text{ even and } 0 \leq j \leq n-2,$$

and

$$\alpha^{(k)}(x) \geq u^{(k)}(x) \geq \beta^{(k)}(x) \text{ for } k \text{ odd and } 1 \leq k \leq n-3.$$

For n odd,

$$\alpha^{(k)}(x) \leq u^{(k)}(x) \leq \beta^{(k)}(x) \text{ for } k \text{ odd and } 1 \leq k \leq n-2,$$

and

$$\alpha^{(j)}(x) \geq u^{(j)}(x) \geq \beta^{(j)}(x) \text{ for } j \text{ even and } 0 \leq j \leq n-3.$$

Therefore, condition (21) holds for $i = 0, \dots, n-2$.

From Lemma 3 and the definition of ξ, the right hand side of equation (19) is a L^1-function. Therefore, Lemma 1 can be applied with $m_j(x) = \gamma_j(x)$ and $M_j(x) = \Gamma_j(x)$ for $j = 0, \dots, n-2$, that is, condition (22) holds.

Step 2: *Problem* (19)–(20) *has a solution* $u_1(x)$.

First we consider the case $n \geq 3$. Let $u \in C^{n-1}(I)$ be fixed. By Lemma 2, solutions of problem (19)–(20) are the fixed points of the operator

$$\begin{aligned}\mathcal{T}\, u(x) = &\sum_{k=0}^{n-3}(-1)^k \delta_k\big(1, u^{(k)}(1) + g_k(u, \dots, u^{(n-2)}, u^{(k)}(1))\big)\frac{(1-x)^k}{k!}\\ &+ (-1)^n \int_x^1 \frac{(s-x)^{n-3}}{(n-3)!} v_u(s)\mathrm{d}s,\end{aligned}$$

with

$$v_u(x) := g_{n-2}\big(u, u', \dots, u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\big) + \int_0^x \varphi^{-1}\left(\tau_u - \int_0^s F_u(r)\mathrm{d}r\right)\mathrm{d}s$$

and $\tau_u \in \mathbb{R}$ is the unique solution of the equation

$$\begin{aligned}(23)\quad &g_{n-1}\big(u, u', \dots, u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\big)\\ &- g_{n-2}\big(u, u', \dots, u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\big) = \int_0^1 \varphi^{-1}\left(\tau_u - \int_0^s F_u(r)\mathrm{d}r\right)\mathrm{d}s.\end{aligned}$$

By (19), there is a function $\omega \in L^1(I)$ such that

$$|F_u(s)| \leq \omega(s) \text{ for a. e. } s \in I \text{ and for all } u \in C^{n-1}(I),$$

and by (23), there exists $L > 0$ such that

$$|\tau_u| \leq L \text{ for all } u \in C^{n-1}(I).$$

Thus, the operator $\mathcal{T}(C^{n-1}(I))$ is bounded in $C^{n-1}(I)$, and by Schauder's fixed point theorem, $\mathcal{T}$ has a fixed point u_1. If $n = 2$, a similar proof holds.

Step 3: $u_1(x)$ *is a solution of problem* (1)–(2).

To see this, it suffices to show that

$$\gamma_i(1) \le u_1^{(i)}(1) + g_i\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(i)}(1)\big) \le \Gamma_i(1), \quad i = 0, \ldots, n-3, \tag{24}$$

$$\alpha^{(n-2)}(0) \le u_1^{(n-2)}(0) + g_{n-2}\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\big) \le \beta^{(n-2)}(0), \tag{25}$$

and

$$\alpha^{(n-2)}(1) \le u_1^{(n-2)}(1) + g_{n-1}\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\big) \le \beta^{(n-2)}(1).$$

Suppose that n is even. Consider the case $i = 0$. Then, by (14), $\gamma_0\,(1) = \alpha(1)$ and $\Gamma_0\,(1) = \beta(1)$. Assume, for the sake of a contradiction, that

$$u_1(1) + g_0\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1(1)\big) > \beta(1).$$

By (20), $u_1(1) = \beta\,(1)$, and by (H_1) and Definition 2, we have

$$0 < g_0\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1(1)\big) = g_0\big(u_1, u_1', \ldots, u_1^{(n-2)}, \beta(1)\big) \le g_0\big(\beta, \beta', \ldots, \beta^{(n-2)}, \beta(1)\big) \le 0,$$

which is a contradiction. Hence,

$$u_1(1) + g_0\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1(1)\big) \le \beta(1).$$

By the same technique, it can be shown that

$$\alpha(1) \le u_1(1) + g_0\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1(1)\big)$$

as well as the remaining inequalities in (24).

Suppose the first inequality in (25) does not hold. Then, from (20), we have $u_1^{(n-2)}(0) = \alpha^{(n-2)}(0)$, and by (21), $u_1^{(n-1)}(0) \ge \alpha^{(n-1)}\,(0)$. By the monotone assumptions on g_{n-2}, (10) yields

$$\begin{aligned} 0 &> g_{n-2}\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\big) \\ &= g_{n-2}\big(u_1, u_1', \ldots, u_1^{(n-2)}, \alpha^{(n-2)}(0), u_1^{(n-1)}(0)\big) \\ &\ge g_{n-2}\big(\alpha, \alpha', \ldots, \alpha^{(n-2)}, \alpha^{(n-2)}(0), \alpha^{(n-1)}(0)\big) \ge 0, \end{aligned}$$

which again is a contradiction. Hence,

$$\alpha^{(n-2)}(0) \le u_1^{(n-2)}(0) + g\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\big).$$

A similar approach shows that

$$u^{(n-2)}(0) + g_{n-2}\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\big) \le \beta^{(n-2)}\,(0)\,.$$

Now assuming that

$$\alpha^{(n-2)}(1) > u_1^{(n-2)}(1) + g_{n-1}\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\big),$$

similar arguments show $u_1^{(n-2)}(1) = \alpha^{(n-2)}(1)$ and $u_1^{(n-1)}(1) \leq \alpha^{(n-1)}(1)$. Therefore, from the properties of g_{n-1}, we have

$$\begin{aligned} 0 &> g_{n-1}\big(u_1, u_1', \ldots, u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\big) \\ &= g_{n-1}\big(u_1, u_1', \ldots, u_1^{(n-2)}, \alpha^{(n-2)}(1), u_1^{(n-1)}(1)\big) \\ &\geq g_{n-1}\big(\alpha, \alpha', \ldots, \alpha^{(n-2)}, \alpha^{(n-2)}(1), \alpha^{(n-1)}(1)\big) \geq 0, \end{aligned}$$

which is a contradiction. The remaining inequality can also be demonstrated by the above technique.

For n odd, the arguments are analogous using the monotone assumptions in (H_2) and the corresponding boundary conditions. □

4. EXAMPLES

In this section, we present two examples to illustrate the cases of n odd and even. The boundary conditions are chosen not for their physical meaning but to emphasize the possibilities available in the functional dependence.

Example 1. For $n = 3$, consider the problem consisting of the equation

$$\frac{u'''(x)}{1 + (u''(x))^2} = (u(x))^3 + k\big(u'(x)\big)^5 - \sqrt[3]{u''(x) + 1} \tag{26}$$

and the boundary conditions

$$\begin{aligned} Au(1) &= \sum_{j=1}^{+\infty} a_j u(\xi_j) - \sum_{j=1}^{+\infty} b_j u'(\eta_j), \\ Bu'(0) &= \max_{x\in[0,1]} u'(x) - \int_0^x u(t)\mathrm{d}t + \big(u''(0)\big)^{2p+1}, \\ Cu'(1) &= \min_{x\in[0,1]} u'(x) - \max_{x\in[0,1]} u(x) - \big(u''(1)\big)^{2q+1}, \end{aligned} \tag{27}$$

with k, A, B, $C \in \mathbb{R}$, $0 \leq \xi_j$, $\eta_j \leq 1$ for all $j \in \mathbb{N}$, p, $q \in \mathbb{N}$, and $\sum_{j=1}^{+\infty} a_j$ and $\sum_{j=1}^{+\infty} b_j$ are nonnegative and convergent series with sums $\overline{a}$ and $\overline{b}$, respectively.

This problem is a particular case of (1)–(2), where $\phi(z) = \arctan z$ (notice that $\phi(\mathbb{R}) \neq \mathbb{R}$),

$$\begin{aligned} f(x, y_0, y_1, y_2) &= -{y_0}^3 - k{y_1}^5 + \sqrt[3]{y_2 + 1}, \\ g_0(z_1, z_2, z_3) &= \sum_{j=1}^{+\infty} a_j z_1(\xi_j) - \sum_{j=1}^{+\infty} b_j z_2(\eta_j) - Az_3, \\ g_1(z_1, z_2, z_3, z_4 &= \max_{x\in[0,1]} z_2 - \int_0^x z_1(t)\mathrm{d}t + z_4^{2p+1} - Bz_3, \\ g_2(z_1, z_2, z_3, z_4) &= \min_{x\in[0,1]} z_2 - \max_{x\in[0,1]} z_1 - z_4^{2q+1} - Cz_3. \end{aligned}$$

The lines $\alpha(x) = 2-x$ and $\beta(x) = x-2$ are, respectively, lower and upper solutions of the problem (26)–(27) for $k \geq 9$, $A \geq 2\overline{a}+\overline{b}$, $B \geq 3$, and $C \geq 3$. Therefore, by Theorem 1, there is a nontrivial solution $u(x)$ of problem (26)–(27), such that

$$\beta(x) = x-2 \leq u(x) \leq 2-x = \alpha(x) \text{ and } \alpha'(x) = -1 \leq u'(x) \leq 1 = \beta'(x),$$

for all $x \in I$.

Example 2. For $n = 4$, consider the functional boundary value problem

$$\begin{aligned}
(u'''(x)^{2p+1})' &= -\arctan(u(x)) + (u'(x))^3 - k(u''(x))^5 - |u'''(x)+1|^\theta, \\
Au(1) &= \max_{x\in[0,1]} u'(x) - \int_0^x u(t)\mathrm{d}t, \\
Bu'(1) &= \sum_{j=1}^{+\infty} a_j\ u''(\xi_j), \\
C\big(u''(0)\big)^3 &= -\max_{x\in[0,1]} u(x-\tau),\ (0<\tau\leq x\leq 1), \\
Du''(1) &= u'(\max\{0, x-\varepsilon\}),\ (\varepsilon > 0)\,,
\end{aligned} \tag{28}$$

where $p \in \mathbb{N}$, $\theta \in [0,2]$, k, A, B, C, $D \in \mathbb{R}$, $0 \leq \xi_j \leq 1$ and $a_j \geq 0$ for $j = 1, 2, \ldots$, and $\sum_{j=1}^{+\infty} a_j$ is convergent with sum $\overline{a}$.

The above problem satisfies the assumptions of Theorem 1 with $\phi(z) = z^{2p+1}$ (in this case $\phi(\mathbb{R}) = \mathbb{R}$),

$$\begin{aligned}
f(x, y_0, y_1, y_2, y_3) &= \arctan y_0 - {y_1}^3 + k{y_2}^5 - |y_3+1|^\theta, \\
g_0(z_1, z_2, z_3, z_4) &= Az_4 - \max_{x\in[0,1]} z_2 + \int_0^x z_1(t)\mathrm{d}t, \\
g_1(z_1, z_2, z_3, z_4) &= Bz_4 - \sum_{j=1}^{+\infty} a_j z_3(\xi_j), \\
g_2(z_1, z_2, z_3, z_4, z_5) &= C{z_5}^3 + \max_{x\in[0,1]} z_1(x-\tau), \\
g_3(z_1, z_2, z_3, z_4, z_5) &= Dz_4 - z_2(\max\{0, x-\varepsilon\}).
\end{aligned}$$

The functions $\alpha(x) = -(2-x)^2$ and $\beta(x) = (2-x)^2$ are, respectively, lower and upper solutions of problem (28) for $k \leq -\frac{\pi}{128} - \frac{9}{32}$, $A \leq -\frac{19}{3}$, $B \leq -\overline{a}$, $C \leq -\frac{1}{8}$, and $D \leq -2$. So, by Theorem 1, there is a nontrivial solution $u(x)$ of problem (28) such that

$$\begin{aligned}
\alpha(x) = -(2-x)^2 \leq u(x) \leq (2-x)^2 = \beta(x), \\
\beta'(x) = 2x-4 \leq u'(x) \leq 4-2x = \alpha'(x),
\end{aligned}$$

and

$$\alpha''(x) = -2 \leq u''(x) \leq 2 = \beta''(x)$$

for all $x \in I$.

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REFERENCES

1. A. Cabada, F. Minhós: *Fully nonlinear fourth order equations with functional boundary conditions.* J. Math. Anal. Appl., **340** (2008), 239–251.
2. A. Cabada, F. Minhós, A. I. Santos: *Solvability for a third order discontinuous fully equation with functional boundary conditions.* J. Math. Anal. Appl., **322** (2006), 735–748.
3. A. Cabada, R. Pouso. F. Minhós: *Extremal solutions to fourth-order functional boundary value problems including multipoint conditions.* Nonlinear Anal. Real World Appl., **10** (2009), 2157–2170.
4. J. R. Graef, L. Kong: *Solutions of second order multi-point boundary value problems.* Math. Proc. Cambridge Philos. Soc., **145** (2008), 489–510.
5. J. R. Graef, L. Kong, Q. Kong: *Higher order multi-point boundary value problems.* Math. Nachr., **284** (2011), 39–52.
6. J. R. Graef, L. Kong, F. Minhós: *Higher order ϕ-Laplacian BVP with generalized Sturm-Liouville boundary conditions.* Differ. Equ. Dyn. Syst., **18** (2010), 373–383.
7. J. R. Graef, L. Kong, F. Minhós: *Higher order functional boundary value problems: existence and location results.* Acta Sci. Math. (Szeged), to appear.
8. J. R. Graef, L. Kong, B. Yang: *Existence of solutions for a higher order multi-point boundary value problems.* Results Math., **53** (2009), 77–101.
9. M. R. Grossinho, F. Minhós, A. I. Santos: *A note on a class of problems for a higher order fully nonlinear equation under one sided Nagumo type condition.* Nonlinear Anal., **70** (2009), 4027–4038.
10. J. Henderson: *Solutions of multipoint boundary value problems for second order equations.* Dynam. Systems Appl., **15** (2006), 111–117.
11. J. Henderson, S. K. Ntouyas: *Positive solutions for systems of n-th order three-point nonlocal boundary value problems.* Electron. J. Qual. Theory Differ. Equ., (18) (2007), 1–12.
12. S. Liang, J. Zhanga: *The method of lower and upper solutions for 2nth-order multi-point boundary value problems.* Nonlinear Anal., **71** (2009), 4581–4587.
13. Y. Liu: *An existence result for solutions of nonlinear Sturm-Liouville boundary value problems for higher order p-Laplacian differential equations.* Rocky Mountain J. Math., **39** (2009), 147–163.
14. F. Minhós: *On some third order nonlinear boundary value problems: Existence, location and multiplicity results.* J. Math. Anal. Appl. **339** (2008), 1342–1353.
15. F. Minhós, J. Fialho: *On the solvability of some fourth-order equations with functional boundary conditions.* Discrete Contin. Dyn. Syst. (2009), suppl., 564–573.
16. H. Pang, W. Ge: *Existence results for some fourth-order multi-point boundary value problem.* Math. Comput. Modelling, **49** (2009), 1319–1325.

17. M. Wang, A. Cabada, J. J. Nieto: *Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions.* Ann. Polon. Math., **58** (1993), 221–235.

18. X. Zhang, L. Liu: *Nontrivial solutions for higher order multi-point boundary value problems.* Comput. Math. Appl., **56** (2008), 861–873.

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