The Hyperbolic Paradigma

Joaquim M. C. Correia
DMat-ECT, CIMA-IIFA, UÉvora & CAMGSD, IST, Portugal
Email: jmcorreia@uevora.pt

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Abstract: As nonlinear hyperbolic partial differential equations have non unique global solutions, I am concerned with two, related, issues: what about physical solutions? and when can we use such a type of equations?

Keywords: hyperbolic conservation law; shock wave; entropy weak solution; measure-valued solution; dissipation; dispersion; diffusion; capilllarity; Burgers equation; KdV-type equation

Hyperbolicity & Real World

$$w_{tt} - (\sigma(w_x))_x = 0\,, \qquad \text{(nonlin. wave eq.)}$$

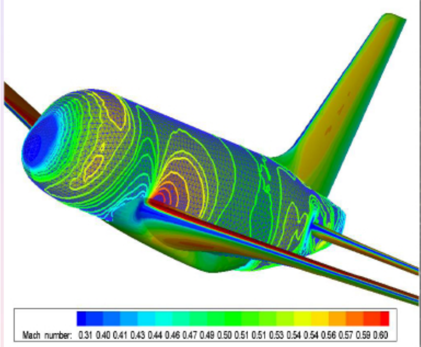
$u := w_t\,,$
$v := w_x\,,$
$p(v) := -\sigma(v)\,,$

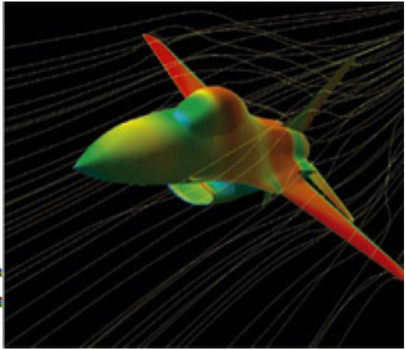
$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \iff \begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_x = 0 \qquad (p\text{-system})$$

Real eigenvalues? $\lambda_1 := -\sqrt{-p'(v)} < \lambda_2 := \sqrt{-p'(v)}$ (speed...)

Nonlinear World:

Discontinuities. . .

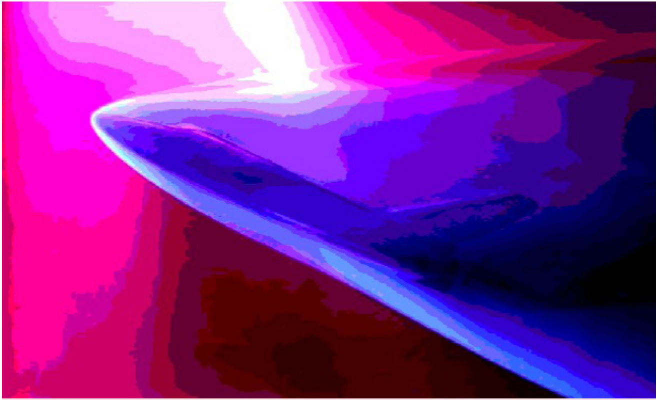




The transonic regime issues:

- control of vibrations and
- shocks strength magnitude

. . . & Irreversibility



NASA: Visible shocks at the nose in the windtunnel test

Conservation Laws:

$$\partial_t u + \mathrm{div}_{\vec{x}} \left(\overbrace{f(u) - \underbrace{\varepsilon\, b(u, \nabla u) - \delta\, \partial_{\color{red}\xi}\, c(u, \nabla u)}_{\text{pertubation} \quad \mathcal{P}_{\varepsilon,\delta}(u;{\color{red}f},b,c)}}^{\text{transport + viscosity + capillarity}} \right) = 0$$

- 'hyperbolic': finite speed of propagation;
- 'divergence form': via modelization of "physical closed systems",
- sources; anisotropy; $\xi \in \{t, x_1, \ldots, x_n\}$:
  - $\xi = t$ (the time): **gBBM-Burgers**;
  - $\xi = x_k$ (one space variable): **gKdV-Burgers**.

Nonlinear Hyperbolic Conservation Laws

- Same simplified equation:

$$\partial_t u + \operatorname{div}_{\vec{x}} f(u) = 0,$$

if we consider the

  - ε,b-viscosity (with diffusive, dissipative effect),
  - δ,c-capillarity (with oscillatory, dispersive effect),

as a

  - "spurious small scale mechanisms",

or at

  - the formal "zero viscosity-capillarity limit" ($\varepsilon, \delta \to 0$):

Singular Limits

N.B. Well-posedness of the (time-evolution) Cauchy problem means that this equation must be hyperbolic and, because it is nonlinear, it develops discontinuities ("shocks") in finite time: the solutions are not unique.

So: how can we select the physically relevant solution?

As the ε,δ-parameters tend to zero and according to the balance of ε,δ-strengths and the growth ratio of *b*-viscosity and *c*-capillarity, we can have:

- *classical-entropy* solutions;
- *nonclassical-entropy* solutions;
- no limit at all.

Paradox

What are the "spurious" $b, c = ???$

A 15 years old conjecture: some "pure capillarity" ($\varepsilon \equiv 0$ or KdV-like) equations have a dissipative behaviour.

Mathematical issues concern:

- the behaviour and selection of the right models/solutions;
- the proof of a "vanishing viscosity-capillarity method".

Physical issues concern:

- Suggestions ?...

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