To my beloved mother for her endless love

and to the loving memory of my father

Analysis and Numerical Simulations of Oldroyd-B Fluids Flows

Abstract

This work is concerned with the mathematical and numerical study of the equations that model incompressible non-Newtonian viscoelastic Oldroyd-B fluids in dimension 2. The constitutive equations for the Oldroyd-B fluids consist of highly nonlinear system of partial differential equations (PDE) of combined elliptic-hyperbolic type. The numerical results are obtained by a technique of decoupling the system into a Navier-Stokes system and a tensorial transport equation.

The study of each problem is divided into three parts:

- the mathematical analysis of the properties of the problem such as existence and uniqueness;
- the numerical analysis with results of existence and uniqueness of approximate solutions as well as a-priori error estimates are established;
- presentation of numerical simulations of two-dimensional benchmark problems.

The purpose of this work is to approximate the solution of the Oldroyd-B problem by the finite elements method, using Hood-Taylor elements for the Navier-Stokes system and discontinuous $\mathbb{P}_1$ elements for the transport equation and to present the results of numerical simulations in two-dimensional case.

Key words: Oldroyd-B fluid model, Navier-Stokes system, transport equations, finite elements method.

Análise e Simulação Numérica de Escoamentos de Fluídos Oldroyd-B

Resumo

Este trabalho tem como objectivo o estudo matemático e numérico das equações que modelam fluidos estacionários não-Newtonianos, viscoelásticos, incompressiveis do tipo Oldroyd-B em dimensão 2. As equações constitutivas para os fluidos do tipo Oldroyd-B consistem num sistema fortemente não linear de equações diferenciais parciais (PDE) do tipo misto elíptico- hiperbólico. Os resultados numéricos são obtidos por uma técnica de desacoplagem do sistema nos problemos Navier-Stokes e equação do transporte.

O estudo de cada problema é dividido em três passos:

- a análise matemática das propriedades das soluções tais como a existência e unicidade;
- a análise numérica com resultados de existência e unicidade das soluções aproximadas e de estimativas de erro a-priori;
- apresentação de simulações numéricas de problemas benchmark bidimensionais.

O objectivo deste trabalho consiste na aproximação da solução do problema do tipo Oldroyd-B pelo método dos elementos finitos, usando elementos de Hood-Taylor para o sistema de Navier-Stokes e elementos $\mathbb{P}_1$ discontinuos para a equação do transporte e apresentar resultados de simulação numérica no caso bidimensional.

Palavras-chave: Modelo de Oldroyd-B, sistema de Navier-Stokes, equação de transporte, métodos de elementos finitos.

Acknowledgement

This thesis would not have been possible without the encouragement, assistance and continuous support of many people. My deepest gratitude to all of them.

I would like to gratefully and sincerely thank my supervisor Prof. Dr. Marília da Conceição Valente de Oliveira Pires, for her wonderful guidance, encouragement, patience and cooperation during the preparation of this thesis and throughout the whole period of study in the University of Évora. Her excellent mentorship provided me with great opportunity to be introduced into the field of numerical methods for Newtonian and non-Newtonian fluids and to gain a well-rounded experience consistent my long-time career goals. Her continuous support encouraged me to grow as an instructor and independent thinker which will be very much helpful during my teaching profession. Thank you, Dr. Marília, for everything you have done for me.

I would like to thank Prof. Dr. Feliz Minhós, Prof. Dr. Clara Carlota, Prof. Dr. Vladimir Goncharov, Prof. Dr. Joaquim Correira, Prof. Dr. António Ornelas and Prof. Dr. Vladimir Bushenkov, for their valuable scientific advice, fruitful discussions and useful guidance during my study and the thesis period.

I would like to thank Prof. Dr. Imme Pieter Van Den Berg, Prof. Dr. José Carlos Brandão Tiago Oliveira, Mrs. Teresa Nogueiro and Mrs. Marta Graça for their assistance, valuable suggestions and all kind of supports.

I would like to express my grateful thanks to Prof. Dr. Md. Shafiqul Islam, Department of Mathematics, University of Dhaka, for his kind support and unending inspiration to study in the abroad.

It is my pleasure to acknowledge my friends and colleagues who have generously offered their help and friendship, and provided some much needed humor and entertainment throughout the study period.

I would like to thank the Department of Mathematics, Univeristy of Évora, especially all the staffs who have provided me all the facilities and cooperation for the completion of the thesis.

In the framework of European Union Program Erasmus Mundus EMMA-WEST Action-2 provided funding for this study. I wish to acknowledge EMMA-WEST project for the financial support during my stay in Portugal.

I would also like to thank Comilla University for allowing me a study leave for the whole period of my study in Portugal.

Finally, and most importantly, I wish to express my deep gratitude towards my family. I am very much thankful to my mother, grandmother and my wife for their sacrifices, constant support and their love. I owe a great deal to them.

Khalifa Mohammad Helal
Évora, Portugal.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The aim of this work is to study the analysis (mathematical and numerical) and numerical simulations, using finite element methods (FEM), of the non-linear system of partial differential equations (PDE) of a combined elliptic-hyperbolic type, that models the non-Newtonian incompressible viscoelastic Oldroyd-B fluids flows in the steady case.

A fluid is a substance which is incapable to prevent the deformation under the action of a shear stress, i.e., a fluid deforms continuously as long as the shear stress is applied. Fluids are divided into two classes: liquids (incompressible fluids) and gases (compressible fluids). The fluid behavior is classified according to the relationship between the shear stress and the shear strain rate or velocity gradient. For a fluid the shear stress is proportional (not necessarily linearly proportional) to the shear strain rate or velocity gradient. If the shear stress is linearly proportional to the shear strain rate or velocity gradient, then this relationship is called Newton's law of viscosity and the proportionality constant is known as dynamic viscosity or viscosity. A fluid which obeys the Newton's law of viscosity is called the Newtonian fluid which can be modeled by the Navier-Stokes equations. Water, air, gasoline, etc. are the examples of Newtonian fluids. On the other hand, the fluid which can not be predicted by the Newtonian model, i.e., which doesn't obey the Newton's law of viscosity is termed as non-Newtonian fluid where the relation between the Cauchy stress and strain rate tensor is non-linear. Fluids with complex microstructure such as biological fluids (e.g., blood), polymeric liquids, inks, paints, shampoo, toothpaste, foodstuff (honey, milk, mayonnaise, ketchup etc.), foams, gel, solutions of high-molecular-weight are some examples of non-Newtonian fluids. The viscoelastic fluids sometimes called polymeric fluids is a particular type of non-Newtonian fluids. The viscoelastic fluids have both the viscous (characteristics of fluids) and the elastic[1] (characteristics of solids) properties.

A viscoelastic fluid contains large molecules composed of many small simple chemical units. These molecules are able to changes their configurations. Such changes may be either local rearrangements of the structure or there may be large overall changes of configurations. There is consequently an entire spectrum of time constants (relaxation time) associated with the rates at which such thermally induced configurational changes take place. It is these time constants that give viscoelastic fluids at least a partial memory (they do not have a permanent memory but will be a longest relaxation time).

The constitutive equations provides us to characterize the mechanical behavior of fluid which relates the Cauchy stress tensor with the kinematics of different quantities. The constitutive equations for non-Newtonian viscoelastic fluids consists of highly non-linear system of partial differential equations (PDE) of combined elliptic-hyperbolic or parabolic-hyperbolic type. The Oldroyd-B fluids model is the constitutive model of rate type which is capable to describe the viscoelastic behavior of flows in the polymeric processing. The Oldroyd-B constitutive equations for steady flow are decoupled into two auxiliary problems, namely, the Navier-Stokes like problems for the velocity and pressure (elliptic part of the system) and the steady tensorial transport equation for the extra stress tensor (hyperbolic part of the system). Both the auxiliary problems are studied separately. The iterative Newton-Raphson method is used to obtain the numerical solution of the Navier-Stokes problem which is discretized using $\mathbb{P}_2 - \mathbb{P}_1$ (Hood-Taylor) finite elements. The iterative method based

[1]By elasticity, one usually means the ability of a material to return to some unique original shape. By viscosity, one usually means the fluid flow resistance which a fluid offers when it is subjected to a tangential force.

on the application of a fixed point method is implemented to solve the steady tensorial transport equation which is discretized using the discontinuous Galerkin finite element method.

An outline of the thesis is as follows. Chapter 2 is concerned with the constitutive equations of incompressible non-Newtonian Oldroyd-B fluids flows. First we introduce some fundamental concepts and quantities of fluid mechanics. Then we deduce the differential form of the constitutive equations of incompressible non-Newtonian fluids of Oldroyd-B type which describes the viscoelastic behavior of the fluid.

In chapter 3, after introducing some well known results for spaces of functions and finite element methods, we present the mathematical and numerical analysis for the steady Navier-Stokes equations. Numerical results are presented to validate the Newton-Raphson method and the corresponding code developed in FreeFem++.

Chapter 4 is devoted to the analysis of the steady tensorial transport problem. Mathematical analysis for transport problem is discussed and the discontinuous Galerkin finite element method is applied to solve this transport equations. Numerical results are given.

Chapter 5 is concerned with the steady viscoelastic Oldroyd-B fluid model in a two-dimensional domain. We present the approach and discrete problem of Oldroyd-B model. Based on chapter 2 and chapter 3, the numerical simulations of Oldroyd-B fluids flows problem are discussed. The numerical results for the four-to-one abrupt contraction in a plane domain are analyzed. The behaviors of the solutions are discussed and are compared for different cases.

Finally, in chapter 6, we draw some conclusions of this work.

All the simulations are straightforwardly implemented with the script developed in finite element solver FreeFem++.

In appendix, we discuss some basic results which are used in this work.

Chapter 2

Constitutive Equations

In this chapter we deduce the differential form of the basic equations, which are in fact the governing equations of incompressible non-Newtonian fluids of Oldroyd type. We begin with introducing some fundamental concepts of fluid mechanics of continuous medium. Details can be found, for example in [46, 35, 11, 31, 45].

2.1 Kinematics of Fluids

To derive the partial differential governing equations of fluid motions, we need to introduce some kinematics concepts and quantities. We employ the continuum hypothesis, that is, we assume that a fluid may be treated as a continuous medium or continuum, rather than as a group of discrete molecules. In continuum hypothesis, the underlying molecular structure of a fluid is conveniently ignored and replaced by a limited set of fluid properties, defined at each point in the fluid at every instant. Mathematically, this hypothesis allows the use of differential calculus in the modelling and solution of fluid mechanics properties. Here each fluid particle is considered to be a continuous function of position and time.

2.1.1 The principal reference (frames) of describing the motion of continuous medium

To study the kinematics of fluids, the motion of the continuous medium, two *reference* or *frames* can be used. These are *Lagrangian*, and *Eulerian* descriptions and refer to *individual time-rate of change* and *local time-rate of change* respectively. In the Eulerian frame of continuum fluid mechanics, we describe physical space by means of a coordinate system. Every property of a fluid has a corresponding functional dependence on position and time when represented in the Eulerian description. In Eulerian description, we associate a complete set of fluid properties with every point in the spatial grid defined by a selected coordinate system. In the Lagrangian description, a fluid particle is considered as the fundamental entity. This *frame* deals with the history of each fluid particle, i.e., any fluid particle is selected and is moved on its onward direction observing the changes in velocity, pressure, density, etc., at each point and at each instant. If a ball is thrown through the air, our eyes naturally track the ball. This tracking way is considered as Lagrangian description.
Consider the Euclidean coordinate system $\mathbb{R}^d$ $(d = 2, 3)$. We assume that the motion will take place during a time interval $I = [t_0, t_1] \subset \mathbb{R}^+$. Suppose at the reference initial time $t_0 \geq 0$, the domain occupied by the fluid is Ω_0 called initial or reference configuration and at time $t \in I$ the portion of space occupied by the same fluid is Ω_t called current or spatial configuration. The motion of each fluid particle which is on position $\xi \in \Omega_0$ at initial time t_0 and on position $\mathbf{x} \in \Omega_t$ at time $t \in I$ is described by the family of mappings $\mathcal{L}_t$. Precisely,

$$\begin{aligned} \mathcal{L}_t : \Omega_0 &\longrightarrow \Omega_t \\ \xi &\longrightarrow \mathbf{x} = \mathbf{x}(t, \xi) = \mathcal{L}_t(\xi) \end{aligned}$$

where $\mathcal{L}_t$ is called Lagrangian mapping at time t.
We suppose that $\mathcal{L}_t$ is continuous and invertible on $\overline{\Omega_0}$, with continuous inverse. The position $\mathbf{x} \in \Omega_t$ of a material particle is a function of time and the position $\xi \in \Omega_0$ of the same material particle. We can relate the pairs (t, ξ) and $(t, \mathbf{x})$ which are respectively called the *material* or Lagrangian variables and *spatial* or Eulerian

variables. When we use Eulerian variables as independent variables we focus on a set of specific locations in the flow field. When we use Lagrangian variables as independent variables we focus on the position ξ of a specific fluid particle at the initial time t_0. In fact, we are tracking the trajectory T_ξ describe by the particle during the time interval $[t_0, t]$ which was on position ξ at instant t_0. The trajectory is given by

$$T_\xi = \{(t, \mathbf{x}(t, \xi)) : t \in I\}.$$

Though it is more convenient to work with the Eulerian variables, the basic principles of mechanics are more easily formulated with reference to the moving particles, i.e, in the Lagrangian frame. We will mark with the hat symbol $\widehat{\ }$, a quantity expressed as function of Lagrangian variables, that is, if $f : I \times \Omega_t \longrightarrow \mathbb{R}$ we have the quantity

$$\widehat{f}(t, \xi) = f(t, \mathbf{x}), \qquad \text{with} \qquad \mathbf{x} \in \mathcal{L}_t(\xi).$$

To indicate the gradient with respect to the Eulerian variable $\mathbf{x}$ we use the symbol ∇. Gradient with respect to Lagrangian variable ξ is indicated by the symbol ∇_ξ, defined by

$$\nabla_\xi \widehat{f} = \sum_{i=1}^{3} \frac{\partial \widehat{f}}{\partial \xi_i} e_i.$$

For the other differential operators such as divergence, Laplacian, etc., we use same convention.

2.1.2 The fluid velocity

The fluid velocity is the fundamental variable in fluid dynamics. It is the major kinematic quantity. In the Lagrangian frame it is expressed by means of a vector field $\widehat{\mathbf{u}} \equiv \widehat{\mathbf{u}}(t, \xi)$ which is defined as

$$\widehat{\mathbf{u}} = \frac{\partial \mathbf{x}}{\partial t} \ \text{ i.e } \ \widehat{\mathbf{u}}(t, \xi) = \frac{\partial \mathbf{x}}{\partial t}(t, \xi)$$

$\widehat{\mathbf{u}}$ is called the Lagrangian velocity field and it denotes the time derivative along the trajectory T_ξ of the fluid particle which was located at position ξ at the reference time.

For $(t, \mathbf{x}) \in I \times \Omega_t$, the velocity in the Eulerian frame is defined as

$$\mathbf{u} = \widehat{\mathbf{u}} \circ \mathcal{L}_t^{-1} \quad \text{i. e.} \quad \mathbf{u}(t, \mathbf{x}) = \widehat{\mathbf{u}}(t, \mathcal{L}_t^{-1}(\mathbf{x})).$$

In general the velocity field is a three-dimensional or two-dimensional time dependent vector field.

2.1.3 The substantial or material derivative

The derivative of a vector field with respect to a fixed position in space is called a Eulerian derivative. On the other hand the derivative of a vector field following a moving particle of fluid along its path is called substantial, material, co-moving or Lagrangian derivative. This derivative relates the time derivatives computed with respect to the Lagrangian and Eulerian frames. The *material* or Lagrangian time derivative of a function f, which is denoted by $\frac{Df}{Dt}$, is defined as the time derivative in the Lagrangian frame. It is expressed as function in the Eulerian variables.

If f be a mapping such that

$$f : I \times \Omega_t \longrightarrow \mathbb{R} \qquad \text{and} \qquad \widehat{f} = f \circ \mathcal{L}_t$$

then

$$\frac{Df}{Dt} : I \times \Omega_t \longrightarrow \mathbb{R}, \qquad \frac{Df}{Dt}(t, \mathbf{x}) = \frac{\partial \widehat{f}}{\partial t}(t, \xi), \qquad \xi = \mathcal{L}_t^{-1}(\mathbf{x})$$

So, for any fixed $\xi \in \Omega_0$ we can write

$$\frac{Df}{Dt}(t, \mathbf{x}) = \frac{d}{dt} f(t, \mathbf{x}(t, \xi))$$

We can observe that the material derivative represents the rate of variation of f along the trajectory T_ξ. Applying the chain-rule of derivation of composed functions we can write

$$\begin{aligned} \frac{Df}{Dt}(t, \mathbf{x}) &= \frac{d}{dt} f(t, \mathbf{x}(t, \xi)) = \left[\frac{\partial}{\partial t}(f \circ \mathcal{L}_t) \right] \circ \mathcal{L}_t^{-1} = \\ &= \frac{\partial f}{\partial t}(t, \mathbf{x}) + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(t, \mathbf{x}) \cdot \frac{\partial x_i}{\partial t} = \frac{\partial f}{\partial t}(t, \mathbf{x}) + \sum_{i=1}^{d} u_i(t, \mathbf{x}) \frac{\partial f}{\partial x_i}(t, \mathbf{x}) \end{aligned}$$

So, material derivative operator is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

The term $\frac{\partial}{\partial t}$ is a partial time derivative, and the term $\mathbf{u} \cdot \nabla$, called the convective derivative, involves partial space derivatives.

2.1.4 The acceleration of fluid

The fluid acceleration is a kinematic quantity. In the Lagrangian frame the acceleration $\widehat{\mathbf{a}}(t, \xi)$ is a vector field $\widehat{\mathbf{a}} : I \times \Omega_0 \longrightarrow \mathbb{R}$ defined by

$$\widehat{\mathbf{a}}(t, \xi) = \frac{\partial \widehat{\mathbf{u}}}{\partial t}(t, \xi) = \frac{\partial^2 \mathbf{x}}{\partial t^2}(t, \xi)$$

If we use the definition of material derivative, we can write the acceleration in Eulerian frame as

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

We observe that the total acceleration at a point in a fluid can be written as the sum of two different types of acceleration called the local and convective accelerations. The components of the acceleration in Cartesian coordinates can be written as

$$a_i = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j}, \quad i = 1, \ldots, d.$$

2.1.5 The deformation gradient tensor

The deformation gradient tensor is the kinematic quantity necessary for the derivation of the mathematical model in fluid dynamics. The deformation gradient tensor $\widehat{F}_t$, which is defined, at each $t \in I$, as

$$\widehat{F}_t : \Omega_0 \longrightarrow \mathbb{R}^{d \times d}, \quad \widehat{F}_t(\xi) = \nabla_\xi \mathcal{L}_t(\xi) = \frac{\partial \mathbf{x}}{\partial \xi}(t, \xi)$$

where $\nabla_\xi \mathcal{L}_t$ is the derivative of $\mathcal{L}_t$ with respect to Lagrangian variable ξ.

We can write componentwise

$$\left[\widehat{F}_t\right]_{ij} = \frac{\partial x_i}{\partial \xi_j}, \quad i, j = 1, \dots, d.$$

The Jacobian of the mapping $\mathcal{L}_t$ is the determinant $\widehat{J}_t = \det \widehat{F}_t > 0$. In the Eulerian frame its counterpart is indicated by J_t.

Using the determinant of deformation gradient tensor we can transform integrals from the current to the reference configuration. The next theorem [35, 11] tells us about the transformation.

Theorem 2.1.1

Suppose $V_t \subset \Omega_t$ be a subdomain of Ω_t and let us consider the function $\widehat{f} : I \times V_t \longrightarrow \mathbb{R}$. Then, f is integrable on V_t if and only if $(f \circ \mathcal{L}_t)J_t$ is integrable on $V_0 = \mathcal{L}_t^{-1}(V_t)$, and

$$\int_{V_t} f(t, \mathbf{x}) d\mathbf{x} = \int_{V_0} \widehat{f}(t, \xi) J_t(\xi) d\xi.$$

where $\widehat{f}(t, \xi) = f(t, \mathcal{L}_t(\xi))$. Briefly

$$\int_{V_t} f = \int_{V_0} \widehat{f} J_t.$$

□

The next lemma tells us that the time derivative of the Jacobian is linked to the divergence of the velocity field. This relation is called Euler expansion formula. Its proof can be found in [35, 11].

Lemma 2.1.1

Let J_t denote the Jacobian in the Eulerian frame. Then

$$\frac{\partial}{\partial t} J_t(\mathbf{x}) = \frac{\partial}{\partial t} J(t, \mathbf{x}) = J_t(\mathbf{x}) \nabla \cdot \mathbf{u}(t, \mathbf{x}) = J(t, \mathbf{x}) \nabla \cdot \mathbf{u}(t, \mathcal{L}_t(\xi)). \tag{2.1}$$

□

Theorem 2.1.2 *(Reynolds Transport Theorem)*

Let $V_0 \subset \Omega_0$ and $V_t \subset \Omega_t$ be its image under the mapping $\mathcal{L}_t$. Let $f : I \times \Omega_t \longrightarrow \mathbb{R}$ be continuously differentiable with respect to both variables. Then,

$$\frac{d}{dt} \int_{V_t} f = \int_{V_t} \left(\frac{Df}{Dt} + f \, \nabla \cdot \mathbf{u} \right) = \int_{V_t} \left[\frac{\partial f}{\partial t} + \nabla \cdot (f \, \mathbf{u}) \right] \tag{2.2}$$

□

The proof of this theorem can be found in [11].

Theorem 2.1.3 *(Divergence Theorem)*
Let Ω is open bounden domain in $\mathbb{R}^d$, $(d = 2, 3)$ with a piecewise smooth boundary $\partial\Omega$. If $\mathbf{u}$ is continuously differentiable vector field on a neighborhood of Ω, then

$$\int_\Omega (\nabla \cdot \mathbf{u}) d\Omega = \oint_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) ds, \tag{2.3}$$

where $\mathbf{n} = (n_1, \ldots, n_d)$ is the unit outward normal vector field to the boundary $\partial\Omega$. In index notation, we can write

$$\int_\Omega \frac{\partial u_i}{\partial x_i} = \int_{\partial\Omega} u_i n_i, \quad i = 1, \ldots, d. \tag{2.4}$$

$\square$

Applying the divergence theorem (theorem 2.1.3), we can rewrite (2.2) as

$$\frac{d}{dt}\int_{V_t} f = \int_{V_t} \frac{\partial f}{\partial t} + \int_{\partial V_t} f\mathbf{u}.\mathbf{n}.$$

2.1.6 The rate of deformation and the rate of vorticity tensors

We define the rate of deformation tensor or strain rate tensor by

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right] \tag{2.5}$$

and the rate of vorticity tensor by

$$\mathbf{W}(\mathbf{u}) = \frac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^t\right]. \tag{2.6}$$

Here $\mathbf{D}(\mathbf{u})$ is the symmetric part of the velocity gradient and $\mathbf{W}(\mathbf{u})$ is the antisymmetric part. The rate of deformation gives us information about the rate of change of volume element along the time without rotation effects.
Componentwise,

$$[\mathbf{D}(\mathbf{u})]_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d \ (d = 2, 3).$$

and

$$[\mathbf{W}(\mathbf{u})]_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d \ (d = 2, 3).$$

2.2 Conservation Laws for a Continuum Medium

Conservation laws state the physical principles governing the fluid motion in a continuum medium. According to the conservation laws, a particular measurable property of an isolated physical system does not change as the system evolves. Lavoisier states that "in nature nothing is created, nothing is lost, everything is transformed". The mathematical formulations of these conservation laws are given below.

2.2.1 Conservation of mass or continuity equation

Conservation of mass is a fundamental principle of classical mechanics governing the behavior of a continuum medium. It states that in a fixed region, the total time rate of change of mass is identically zero, i.e, mass is neither created nor destroyed during the motion. Physically, this interprets that the rate of change of the density of a fluid in motion is equal to the sum of the fluid convected into and out of the fixed region. Suppose V_t indicates a material volume at time t, i.e. V_t is the image under the Lagrangian mapping of $V_0 \in \Omega_0$, i.e. $V_t = \mathcal{L}_t(V_0)$. If m_0 is the mass of material in V_0 and m_t is the mass of that material in V_t, then according to the conservation of mass we can say $m_0 = m_t$. Mathematically,

$$m_0 = m(V_0) = m(\mathcal{L}_t(V_0)) = m(V_t) = m_t$$

For each time t, we suppose that the fluid has well-defined mass density ρ (mass per unit volume of material $[\rho] = kg/m^3$) which is strictly positive, measurable function $\rho : I \times \Omega_t \longrightarrow \mathbb{R}$ such that on each $V_t \subset \Omega_t$

$$m\,(V_t) = \int_{V_t} \rho$$

If V_t is a fixed region in Ω_t, then with the mathematical statement of conservation of mass we can write

$$0 = \frac{d}{dt} m\,(V_t) = \frac{d}{dt} \int_{V_t} \rho.$$

Applying the Reynold Transport theorem (theorem 2.1.2), we obtain the integral form of the law of conservation of mass

$$\int_{V_t} \left(\frac{D\rho}{Dt} + \rho \nabla . \mathbf{u} \right) = 0 \Leftrightarrow \int_{V_t} \left(\frac{\partial \rho}{\partial t} + \nabla . \left(\rho \mathbf{u} \right) \right) = 0. \tag{2.7}$$

We suppose that the terms under the integral are continuous. Since the volume V_t is arbitrary, (2.7) is equivalent to the differential equation of this law, called continuity equation (expressing conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla . \left(\rho \mathbf{u} \right) = 0. \tag{2.8}$$

If the density is constant or its material derivative $\frac{D\rho}{Dt} = 0$, from (2.8) the equation of continuity is simplified to

$$\nabla . \mathbf{u} = 0 \tag{2.9}$$

The above relation in the case of incompressible fluid, is in fact a kinematic constraint. Using Euler expansion formula (2.1) we can write the above equation as

$$\frac{\partial}{\partial t} J_t = 0$$

which is incompressibility constraint. A flow satisfying the incompressibility constraint is called *incompressible flow*. By the continuity equation we get the following implication:

constant density fluid$\Rightarrow$ incompressible flow

but the converse is not always true. Mathematically, we mean that the velocity field of an incompressible flow is divergence free.

2.2.2 Conservation of momentum

The conservation law of momentum for a continuum medium is the extension of the famous Newton's second law of motion, "force = mass × acceleration". For a moving flow field this law describes that the total time rate of change of linear momentum or acceleration of a fluid element is equal to the sum of externally applied forces on a fixed region.

For any continuum, forces acting on a piece of material inside Ω_t are three types.

- *External or Body forces.*

  Body forces are long-range forces whose magnitudes are proportional to the mass. They are external forces act on a fluid, but are not applied by a fluid. Body forces are represented by a vector field $\mathbf{f} : I \times \Omega_t \longrightarrow \mathbb{R}^d$ called *specific body force.* Its dimension unit is $Ne/kg = m/s^2$ as like an acceleration. The body force acting on fluid of volume V_t is given by

$$\int_{V_t} \rho \mathbf{f}.$$

  Gravity force and electromagnetic forces are the familiar examples of body forces.

- *Surface forces or Forces of Stress.*

  Surface forces are short-range forces that act on a fluid element through physical contact between the element and its surroundings. Surface forces represent that part of forces which are imposed on the media through its surface. The magnitude of a surface force is proportional to the contact area between the fluid and its surroundings. Surface forces act on a fluid, and also are applied by a fluid to its surroundings. We suppose that the surface force can be represented through a vector field $\mathbf{t}^e : I \times \Gamma_t \longrightarrow \mathbb{R}^d$, called *applied stresses*, defined on a measurable subset of the domain boundary $\Gamma_t \subset \partial\Omega_t$. The resultant force acting through the surface is given by

$$\int_{\Gamma_t} \mathbf{t}^e.$$

- *Internal "continuity" forces.*

  The forces that the continuum media particles exert on each other are the internal continuity forces. They are responsible for maintaining material continuity during the movement.

  We recall *Principle of Cauchy*, to model the internal continuity force [11].

Theorem 2.2.1 *(The Cauchy Principle)*

There exists a vector field, $\mathbf{t} : I \times \Omega_t \times S_1 \longrightarrow \mathbb{R}^d$*, called the* Cauchy stress *with*

$S_1 = \left\{ \mathbf{n} \in \mathbb{R}^d : \|\mathbf{n}\| = 1 \right\}$ $(d = 2, 3)$ *such that its integral on the surface of any material domain* $V_t \subset \Omega_t$*, given by*

$$\int_{\partial V_t} \mathbf{t}\left(t, \mathbf{x}, \mathbf{n}\right) ds,$$

is equivalent to the resultant of the material continuity forces acting on V_t*. Here* $\mathbf{n}$ *is the outward normal of* ∂V_t *and* ds *is the area element.* □

This principle states that the only dependence of the internal forces on the geometry of ∂V_t is through $\mathbf{n}$.

We also have

$$\mathbf{t} = \mathbf{t}^e \text{ on } \partial V_t \cap \Gamma_t.$$

Now, we can write the law of conservation of linear momentum. The momentum of the mass at time t of the volume V_t known as *linear momentum* is defined by

$$\int_{V_t} \rho \mathbf{u}$$

For any $t \in I$ and $V_t \subset \Omega_t$ completely contained in Ω_t ,

$$\frac{d}{dt} \int_{V_t} \rho(t, \mathbf{x}) \mathbf{u}(t, \mathbf{x}) d\mathbf{x} = \int_{V_t} \rho(t, \mathbf{x}) \mathbf{f}(t, \mathbf{x}) d\mathbf{x} + \int_{\partial V_t} \mathbf{t}(t, \mathbf{x}, \mathbf{n}) ds, \tag{2.10}$$

The equation (2.10) tells us the property that the variation of the linear momentum of V_t is balanced by the resultant of the internal and the body forces.

With the following *Cauchy Stress Tensor theorem* we can relate the internal continuity forces to a tensor field assuming some regularity of the Cauchy stresses. The proof can be found in [35, 45].

Theorem 2.2.2 *(Cauchy Stress Tensor Theorem)*

Suppose that for all $t \in I$*, the body forces* $\mathbf{f}$*, the density* ρ *and the fluid acceleration* $\dfrac{D\mathbf{u}}{Dt}$ *are all bounded functions on* Ω_t*, and let the Cauchy stress vector field* $\mathbf{t}$ *is continuously differentiable with respect to the variable* $\mathbf{x}$ *for each* $\mathbf{n} \in S_1$*, where* S_1 *is the set* $\left\{ \mathbf{n} \in \mathbb{R}^d : \|\mathbf{n}\| = 1 \right\}$ $(d = 2, 3)$ *and continuous with respect to* $\mathbf{n}$*. Then, there exists a continuously differentiable symmetric tensor field, called* Cauchy stress tensor

$$\mathbf{T} : I \times \overline{\Omega_t} \longrightarrow \mathbb{R}^{d \times d}$$

such that

$$\mathbf{t}(t, \mathbf{x}, \mathbf{n}) = \mathbf{T}(t, \mathbf{x}) \cdot \mathbf{n}, \qquad \forall t \in I, \forall \mathbf{x} \in \Omega_t, \forall \mathbf{n} \in S_1.$$

□

With the hypothesis of the Cauchy stress tensor theorem, we have

$$\mathbf{T} \cdot \mathbf{n} = \mathbf{t} = \mathbf{t}^e \text{ on } \partial V_t \cap \Gamma_t. \tag{2.11}$$

and the resultant of the internal forces on V_t is expressed by $\mathbf{T} \cdot \mathbf{n}$. So, we can write

$$\int_{\partial V_t} \mathbf{T} \cdot \mathbf{n} = \int_{\partial V_t} \mathbf{t}^e. \tag{2.12}$$

The stress tensor $\mathbf{T}$ represents the forces which the material develops in response to being deformed. Using the above results we can rewrite the law of linear momentum as follows:
For any $t \in I$, the following relation[1] holds on any sub-domain $V_t \subset \Omega_t$

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{u} = \int_{V_t} \rho \mathbf{f} + \int_{\partial V_t} \mathbf{T} \cdot \mathbf{n}. \tag{2.14}$$

Since ρ is constant and as a result $\nabla \cdot \mathbf{u} = 0$, by using the Transport theorem (theorem 2.1.2) we get

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{u} = \int_{V_t} \left[\frac{D(\rho \mathbf{u})}{Dt} + \rho \mathbf{u} \nabla \cdot \mathbf{u} \right] = \int_{V_t} \rho \frac{D(\mathbf{u})}{Dt}. \tag{2.15}$$

So the relation (2.14) can be written as

$$\int_{V_t} \rho \frac{D\mathbf{u}}{Dt} = \int_{V_t} \rho \mathbf{f} + \int_{\partial V_t} \mathbf{T} \cdot \mathbf{n}. \tag{2.16}$$

Applying the divergence theorem (theorem 2.1.3) and assuming that $\nabla \cdot \mathbf{T}$ is integrable, the above relation (2.16) becomes

$$\int_{V_t} \rho \frac{D\mathbf{u}}{Dt} = \int_{V_t} \rho \mathbf{f} + \int_{V_t} \nabla \cdot \mathbf{T}, \tag{2.17}$$

[1] If V_t has a part of boundary in common with Γ_t, then we should use

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{u} = \int_{V_t} \rho \mathbf{f} + \int_{\partial V_t \setminus \Gamma_t} \mathbf{T} \cdot \mathbf{n} + \int_{\partial V_t \cap \Gamma_t} \mathbf{t}^e. \tag{2.13}$$

.

which implies

$$\int_{V_t} \left(\rho \frac{D\mathbf{u}}{Dt} - \nabla \cdot \mathbf{T} - \rho \mathbf{f} \right) = 0. \tag{2.18}$$

Since the volume V_t is arbitrary, with the hypothesis that the terms under the integrals are continuous in space, we derive the differential form of principle of linear momentum

$$\rho \frac{D\mathbf{u}}{Dt} - \nabla \cdot \mathbf{T} = \rho \mathbf{f} \quad \text{in } \Omega_t. \tag{2.19}$$

Writing the fluid acceleration $\dfrac{D\mathbf{u}}{Dt} = \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$, the relation (2.19) finally can be written as

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbf{T} = \rho \mathbf{f}. \tag{2.20}$$

Componentwise

$$\rho \frac{\partial u_i}{\partial t} + \rho \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} - \sum_{j=1}^{d} \frac{\partial T_{ij}}{\partial x_j} = \rho f_i \qquad i = 1, \ldots, d.$$

The non linear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is called the *convective term.*

2.3 Formulation of the Constitutive Relations

All materials mostly satisfy the fundamental conservation principles stated above. The mathematical specification of 'material response' laws is said to be the set of constitutive relations. This law relates the Cauchy stress tensor with the kinematics of different quantities, in particular, the velocity field. Constitutive relations provides us to characterize the mechanical behavior of fluid. In this work we are concerned with non-Newtonian fluids type, in particular with the flows of incompressible viscoelastic Oldroyd-B fluids. We first give the general form of constitutive equations and then we give the overview of differential constitutive equations for viscoelastic fluids of Oldroyd-B having properties of elastic solids and viscous fluids characterized by a viscous behavior when subject to slow request and elastic behavior subjected to fast request.

2.3.1 Principles of formulating the constitutive equations

We take into account several principles and assumptions to formulate a constitutive equation.

- Principle of determinism: We can determine the stress only by history and present state of material.
- Principle of material objectivity: The structure of constitutive equation is independent of the motion of an observer.

We assume that the stress at a material point is determined by the deformation gradient at this point, i.e., we assume the material is simple fluid.
Under the above principles, for simple, isotropic, incompressible fluid, the Cauchy stress tensor $\mathbf{T}$ can be expressed as

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}_s$$

where p is the hydrostatic pressure[2], $\boldsymbol{\tau}_s$ is the extra stress tensor and I is the identity matrix or Kronecker tensor.

2.3.2 Non-Newtonian fluids

If for a fluid, the dissipative effects of frictional forces can be described by a linear relation between the extra stress tensor and rate of strain tensor, i.e,

$$\boldsymbol{\tau}_s = 2\mu\mathbf{D}(\mathbf{u}) \quad \text{(Stokes law)} \tag{2.21}$$

then this fluid is called Newtonian fluid. In (2.21), μ is the dynamic viscosity coefficient expressing the fluid's resistance which it offers to shear strain during the displacement ($[\mu] = Pa\, s$).

[2]Hydrostatic pressure also known as gravitational pressure is the pressure exerted by a fluid at equilibrium at a point in the stationary fluid, due to the weight (force of gravity) of the fluid above it. Hydrostatic pressure (varies with height), is the negative of the stress normal to a surface in the fluid.

On the other hand, the fluids for which the relation between the Cauchy stress tensor and the strain rate tensor is non-linear (doesn't obey the Stokes law) are called non-Newtonian fluids. The fluids with complex microstructures such as polymeric liquids, foams, inks, magma or biological fluids are some examples of non-Newtonian fluids. They are characterized by the fact that they exhibit at least one behavior such as shear-thinning or shear-thickening, stress-relaxation, non-linear creep, normal stress differences or yielding. Some properties of non-Newtonian fluids:

- The non-Newtonian fluid has the ability to thinning and thickening by the action of shear or tangential stress forces, i.e. it has the ability to become more or less viscous as the shear rate increases. In a Newtonian fluids, the viscosity remains constant in time.

- The non-Newtonian fluids deform by the presence of constant tensions, with strain rate is not constant in time. But the Newtonian fluid does not deform under the presence of constant tensions.

- In contrast to Newtonian fluids, some non-Newtonian fluids do not relax stress immediately.

- In some non-Newtonian fluids, the normal tensions varies in simple flows (flows with one-dimensional velocity and velocity gradients), generally normal tension increases with shear rate.

- In the presence of threshold tensions (stress of transfer), some non-Newtonian fluids do not flow immediately, they resist until a certain value of tension.

2.3.3 Models of viscoelastic fluids of Oldroyd type

The constitutive equation in differential form or integral form is suitable to use in a numerical simulation. Oldroyd observed that the convected time derivative $\frac{D\boldsymbol{\pi}}{Dt}=\frac{\partial\boldsymbol{\pi}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\pi}$ of a tensor $\boldsymbol{\pi}$ is not the objective. The objective form of the time derivative of a tensor can be expressed as

$$\frac{\mathcal{D}_a\boldsymbol{\pi}}{\mathcal{D}t}=\frac{d\boldsymbol{\pi}}{dt}+\boldsymbol{\pi}\mathbf{W}(\mathbf{u})+(\boldsymbol{\pi}\mathbf{W}(\mathbf{u}))^t-a\left[\boldsymbol{\pi}\mathbf{D}(\mathbf{u})+(\boldsymbol{\pi}\mathbf{D}(\mathbf{u}))^t\right]$$

i.e.,

$$\frac{\mathcal{D}_a \boldsymbol{\pi}}{\mathcal{D}t} = \frac{\partial \boldsymbol{\pi}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\pi} + \boldsymbol{\pi}\mathbf{W}(\mathbf{u}) - \mathbf{W}(\mathbf{u})\boldsymbol{\pi} - a\left[\boldsymbol{\pi}\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\boldsymbol{\pi}\right] \tag{2.22}$$

where $-1 \le a \le 1$ is a parameter.

The case with $a = -1$, $a = 1$ and $a = 0$ are respectively called lower, upper and co-rotational convected time derivative. Oldroyd suggested a general form of constitutive equation as [23]

$$\lambda_1 \frac{\mathcal{D}_a \boldsymbol{\tau}_s}{\mathcal{D}t} + \boldsymbol{\tau}_s + \gamma(\boldsymbol{\tau}_s, \nabla\mathbf{u}) = 2\mu\left[\lambda_2 \frac{\mathcal{D}_a \mathbf{D}(\mathbf{u})}{\mathcal{D}t} + \mathbf{D}(\mathbf{u})\right], \quad 0 \le \lambda_2 < \lambda_1 \tag{2.23}$$

where the tensor $\boldsymbol{\tau}_s$ is the extra stress, μ is the dynamic viscosity coefficient of fluid which is assumed to be constant and positive, $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$ are the constants depend on the continuous medium, respectively, called the relaxation and retardation time of fluid. λ_1 characterizes the time it takes the fluid to decrease the tension after have been applied a constant deformation and λ_2 characterizes the time it takes the fluid to decrease their state of deformation after having an applied tension. $\gamma(\boldsymbol{\tau}_s, \nabla\mathbf{u})$ is a tensor defined by the traces of $\boldsymbol{\tau}_s$ and /or $\mathbf{D}(\mathbf{u})$. There are several types of general model. Here we write some models with $\gamma(\boldsymbol{\tau}_s, \nabla\mathbf{u}) = 0$.

- Maxwell type fluid models ($\lambda_2 = 0$).
- Jeffreys type fluid models ($\lambda_2 \neq 0$).
- Oldroyd-A fluid ($\lambda_1 > \lambda_2 > 0$ and $a = -1$).
- Oldroyd-B fluid ($\lambda_1 > \lambda_2 > 0$ and $a = 1$).

We can generalize these models. For example, the extra-stress $\boldsymbol{\tau}_s$ can be written as a sum of partial stresses $\boldsymbol{\tau}_s = \sum \boldsymbol{\tau}_s^i$. For each partial stresses $\boldsymbol{\tau}_s^i$ there is a constitutive equation with different relaxation time λ_1^i.

2.3.4 Models of the fluids of Oldroyd-B type

The Cauchy stress tensor is given by

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}_s$$

In viscoelastic fluids, the stresses depend not only on the current motion of the fluid, but on the history of the motion. We can say that λ_1 and λ_2 are the measures of the time for which the fluid remembers the flow history.
Decomposing the extra-stress tensor $\boldsymbol{\tau}_s$ into the sum of its Newtonian part $\boldsymbol{\sigma}_n$ and its viscoelastic part $\boldsymbol{\sigma}_e$, we can write

$$\boldsymbol{\tau}_s = \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e$$

where $\boldsymbol{\sigma}_n = 2\mu\frac{\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{u})$, with $\mu_n = \mu\frac{\lambda_2}{\lambda_1}$ the coefficient of Newtonian viscosity.
Therefore, the Cauchy stress tensor can be written as

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e = -p\mathbf{I} + 2\mu\frac{\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{u}) + \boldsymbol{\sigma}_e \tag{2.24}$$

From (2.23), for Oldroyd-B fluid, i.e., for $\gamma(\boldsymbol{\tau}_s, \nabla\mathbf{u}) = 0,\ \lambda_1 > \lambda_2 > 0$ and $a = 1$, the general form of the constitutive equation can be written as

$$\lambda_1\frac{\mathcal{D}_a\boldsymbol{\tau}_s}{\mathcal{D}t} + \boldsymbol{\tau}_s = 2\mu\left[\lambda_2\frac{\mathcal{D}_a\mathbf{D}(\mathbf{u})}{\mathcal{D}t} + \mathbf{D}(\mathbf{u})\right] \Leftrightarrow$$

$$\lambda_1\frac{\mathcal{D}_a(\boldsymbol{\sigma}_e + \boldsymbol{\sigma}_n)}{\mathcal{D}t} + \boldsymbol{\sigma}_e + \boldsymbol{\sigma}_n = 2\mu\left[\lambda_2\frac{\mathcal{D}_a\mathbf{D}(\mathbf{u})}{\mathcal{D}t} + \mathbf{D}(\mathbf{u})\right] \Leftrightarrow$$

$$\lambda_1\frac{\mathcal{D}_a\boldsymbol{\sigma}_e}{\mathcal{D}t} + \lambda_1 2\mu\frac{\lambda_2}{\lambda_1}\frac{\mathcal{D}_a\mathbf{D}(\mathbf{u})}{\mathcal{D}t} + \boldsymbol{\sigma}_e + 2\mu\frac{\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{u}) = 2\mu\left[\lambda_2\frac{\mathcal{D}_a\mathbf{D}(\mathbf{u})}{\mathcal{D}t} + \mathbf{D}(\mathbf{u})\right] \Leftrightarrow$$

$$\begin{aligned}\lambda_1\frac{\mathcal{D}_a\boldsymbol{\sigma}_e}{\mathcal{D}t} + \boldsymbol{\sigma}_e &= 2\mu\left(1 - \frac{\lambda_2}{\lambda_1}\right)\mathbf{D}(\mathbf{u}) \\ &= 2(\mu - \mu_n)\mathbf{D}(\mathbf{u}) \\ &= 2\mu_e\mathbf{D}(\mathbf{u})\end{aligned}$$

where $\mu_e = \mu - \mu_n$ is the coefficient of elastic viscosity and $\mu = \mu_e + \mu_n$.
So, we have

$$\lambda_1\frac{\mathcal{D}_a\boldsymbol{\sigma}_e}{\mathcal{D}t} + \boldsymbol{\sigma}_e = 2\mu_e\mathbf{D}(\mathbf{u}) \tag{2.25}$$

Finally we can write by (2.22)

$$\lambda_1\left[\frac{\partial\boldsymbol{\sigma}_e}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}_e\right] + \boldsymbol{\sigma}_e = 2\mu_e\mathbf{D}(\mathbf{u}) - \lambda_1[\boldsymbol{\sigma}_e\mathbf{W}(\mathbf{u}) - \mathbf{W}(\mathbf{u})\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_e\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u})\boldsymbol{\sigma}_e] \tag{2.26}$$

Taking into account (2.24), the conservation law of momentum (2.20) can be written as follows

$$\begin{aligned}
\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u} &= \nabla\cdot\mathbf{T}+\rho\mathbf{f}\\
&= \nabla\cdot[-p\mathbf{I}+2\mu_n\mathbf{D}(\mathbf{u})+\boldsymbol{\sigma}_e]+\rho\mathbf{f}\\
&= \nabla\cdot(-p\mathbf{I})+\nabla\cdot\left[\mu_n\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^t\right]\right]+\nabla\cdot\boldsymbol{\sigma}_e+\rho\mathbf{f}\\
&= -p\nabla\cdot\mathbf{I}-\nabla p\cdot\mathbf{I}+\mu_n\nabla\cdot\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^t\right]+\nabla\cdot\boldsymbol{\sigma}_e+\rho\mathbf{f}\\
&= -\nabla p\cdot\mathbf{I}+\mu_n\nabla\cdot(2\mathbf{D}(\mathbf{u}))+\nabla\cdot\boldsymbol{\sigma}_e+\rho\mathbf{f}\\
&= -\nabla p+2\mu_n\nabla\cdot\mathbf{D}(\mathbf{u})+\nabla\cdot\boldsymbol{\sigma}_e+\rho\mathbf{f} \qquad (2.27)
\end{aligned}$$

If $\nabla\cdot\mathbf{u}=0$, then (see appendix $(A-17)$)

$$2\nabla\cdot\mathbf{D}\left(\mathbf{u}\right)=\Delta\mathbf{u}.$$

So, we can also write the conservation of momentum as

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\mu_n\Delta\mathbf{u}+\nabla\cdot\boldsymbol{\sigma}_e+\rho\mathbf{f}$$

For the simplicity, we write $\boldsymbol{\sigma}$ instead of $\boldsymbol{\sigma}_e$. We have the system of non-linear equations formed by the law of conservation of mass (2.9), the momentum equations (2.27) and the Oldroyd-B constitutive equation (2.26) as

$$\begin{cases}
\rho\dfrac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu_n\Delta\mathbf{u}+\nabla p=\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{f}, \quad \text{in } \Omega\\
\nabla\cdot\mathbf{u}=0, \quad \text{in } \Omega\\
\lambda_1\left[\dfrac{\partial\boldsymbol{\sigma}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}\right]+\boldsymbol{\sigma}=2\mu_e\mathbf{D}(\mathbf{u})-\lambda_1[\boldsymbol{\sigma}\mathbf{W}(\mathbf{u})-\mathbf{W}(\mathbf{u})\boldsymbol{\sigma}-\boldsymbol{\sigma}\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u})\boldsymbol{\sigma}]\,,\text{in}\,\Omega.
\end{cases} \qquad (2.28)$$

Assuming

$$\begin{aligned}
\mathbf{h}(\boldsymbol{\sigma},\nabla\mathbf{u}) &= 2\mu_e\mathbf{D}(\mathbf{u})-\lambda_1\left[\boldsymbol{\sigma}\mathbf{W}(\mathbf{u})-\mathbf{W}(\mathbf{u})\boldsymbol{\sigma}-\boldsymbol{\sigma}\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u})\boldsymbol{\sigma}\right]\\
&= 2\mu_e\mathbf{D}(\mathbf{u})+\lambda_1\left[(\nabla\mathbf{u})\boldsymbol{\sigma}+\boldsymbol{\sigma}(\nabla\mathbf{u})^t\right]
\end{aligned}$$

the Oldroyd-B constitutive equations (2.28) can be written as

$$\begin{cases} \rho\dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu_n\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f}, \quad \text{in} \;\; \Omega \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in} \;\; \Omega \\ \lambda_1\left[\dfrac{\partial\boldsymbol{\sigma}}{\partial t} + (\mathbf{u}\cdot\nabla)\sigma\right] + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma},\nabla\mathbf{u}), \quad \text{in} \;\; \Omega. \end{cases} \tag{2.29}$$

The above set of equations describes the behavior of an incompressible viscoelastic fluid of Oldroyd-B type, in a certain open subset Ω of $\mathbf{R}^d$ $(d = 2, 3)$ where the fluid is homogeneous. We observe that the conservation of momentum leads the symmetry properties of the tensor $\boldsymbol{\sigma}$, i.e., $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}$.

The problem (2.29) is a mixed problem. The first two equations form a parabolic system for $(\mathbf{u}, p)$ which is in the form of Navier-stokes equation. The last equation has a hyperbolic characteristic which is in the form of Transport equation.

If the flow state (velocity, pressure, density, etc.,) of a flow does not change with time, then it is called a steady or stationary flow. Therefore, in case of steady flow, $\mathbf{u}$ is independent of time and then $\dfrac{\partial \mathbf{u}}{\partial t} = 0$. So, the Oldroyd-B constitutive equations in case of steady flow is a non-linear system of partial differential equations (PDE) of a combined elliptic-hyperbolic type

$$\begin{cases} \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu_n\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f}, \quad \text{in} \;\; \Omega \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in} \;\; \Omega \\ \lambda_1\,(\mathbf{u}\cdot\nabla)\,\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma},\nabla\mathbf{u}), \quad \text{in} \;\; \Omega. \end{cases} \tag{2.30}$$

2.3.5 Newtonian fluids. Navier-Stokes equations

The limit case $\lambda_1 = 0$ leads us (from (2.25))

$$\boldsymbol{\sigma}_e = 2\mu_e\mathbf{D}(\mathbf{u}).$$

Then the Cauchy stress tensor is given by

$$\begin{aligned}\mathbf{T} &= -p\mathbf{I} + \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e \\ &= -p\mathbf{I} + 2\mu_n \mathbf{D}(\mathbf{u}) + 2\mu_e \mathbf{D}(\mathbf{u}) \\ &= -p\mathbf{I} + 2(\mu_n + \mu_e)\mathbf{D}(\mathbf{u}) \\ &= -p\mathbf{I} + 2\mu \mathbf{D}(\mathbf{u}) \\ &= -p\mathbf{I} + \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right]\end{aligned} \tag{2.31}$$

The Cauchy stress tensor can be written as a linear function of strain rate tensor or the velocity derivative. The fluids for which the above property holds are called the incompressible Newtonian fluids. The Newtonian fluids are a subclass of Stokesian fluids, which are isotropic (with the properties independent of direction) viscous fluids where the stress tensor $\mathbf{T}$ is the sum of the tension caused by the hydrostatic pressure in the fluid, the tension that causes deformation fluid and the tension due to volumetric expansion. Newtonian fluids are modeled by Navier-Stokes equations

$$\begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot\mathbf{T} + \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{T} = -p\mathbf{I} + \mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right] \end{cases} \Longleftrightarrow \begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nabla\cdot\mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right] + \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{2.32}$$

The above system defines the Navier-Stokes equations for incompressible fluids.
For steady flow, the Navier-Stokes equations (2.32) can be written as

$$\begin{cases} \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\Delta\mathbf{u} = \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{2.33}$$

Considering ρ as a constant, we define the kinematic viscosity by $\nu = \dfrac{\mu}{\rho}$ (m^2/s) and the scaled pressure $p = \dfrac{p}{\rho}$ (m^2/s^2) still denoted by p and we obtain from (2.33)

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f} \\ \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{2.34}$$

2.4 Non-dimensional Governing Equations

To obtain a system of dimesionless variables, we discuss some scaling properties of three equations of the system (2.28) to introduce Reynolds number Re and Weissenberg number We that measures the effect of viscosity and elasticity on the flow. Let L be the characteristic length, U represents a characteristic velocity of the flow and $\mu = \mu_n + \mu_e$ be the viscosity coefficient. We transform the system (2.28) into dimensionless form by changing variables and by introducing the following dimensionless quantities:

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad t = \frac{t'}{T} = \frac{Ut'}{L}, \quad \mathbf{u} = \frac{\mathbf{u}'}{U}, \quad p = \frac{p'L}{\mu U}, \quad \boldsymbol{\sigma} = \frac{\boldsymbol{\sigma}'L}{\mu U}, \quad \mathbf{f} = \frac{\mathbf{f}'L^2}{\mu U},$$
$$Re = \rho\frac{UL}{\mu} = \frac{UL}{\nu}, \quad We = \lambda_1\frac{U}{L} \tag{2.35}$$

where the symbol $'$ is attached to dimensional parameters. The dimensionless form of the system (2.28) can be written as

$$\begin{cases} Re\left[\dfrac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] + \nabla p = (1-\lambda)\Delta\mathbf{u} + \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}, \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega \\ We\dfrac{\mathcal{D}_a\boldsymbol{\sigma}}{\mathcal{D}t} + \boldsymbol{\sigma} = 2\lambda\mathbf{D}(\mathbf{u}), \quad \text{in } \Omega. \end{cases} \tag{2.36}$$

where

$$\lambda = 1 - \frac{\lambda_2}{\lambda_1} = \frac{\mu_e}{\mu_e + \mu_n} \tag{2.37}$$

is a retardation parameter.

Reynolds number and Weissenberg number are the dimensionless numbers. Small values of We means that the fluid is little elastic and small values of Re means that the fluid is very viscous. The total stress is given by

$$\boldsymbol{\sigma} + 2(1-\lambda)\mathbf{D}(\mathbf{u})$$

In case of stationary motion, $\dfrac{\partial \mathbf{u}}{\partial t} = 0$. Then the problem can be written as

find $(\mathbf{u}, \boldsymbol{\sigma}, p)$, defined in Ω such that

$$\begin{cases} Re\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right] + \nabla p = (1-\lambda)\Delta\mathbf{u} + \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}, \quad \text{in} \;\; \Omega \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in} \;\; \Omega \\ We\left[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}\right] + \boldsymbol{\sigma} = 2\lambda\mathbf{D}(\mathbf{u}) + We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right], \quad \text{in} \;\; \Omega. \end{cases} \tag{2.38}$$

2.5 Boundary Conditions

Finally, the system (2.30)

$$\begin{cases} \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu_n\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f}, \quad \text{in} \;\; \Omega \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in} \;\; \Omega \\ \lambda_1\left(\mathbf{u}\cdot\nabla\right)\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{h}(\nabla\mathbf{u}, \boldsymbol{\sigma}), \quad \text{in} \;\; \Omega. \end{cases}$$

should be complete with a set of boundary conditions, which depend on the considered geometry.

Taking $\Omega \subset \mathbb{R}^d (d = 2, 3)$ a bounded, simply connected domain, the boundary conditions ensuring feasibility of numerical solution are:

- Dirichlet boundary condition for the velocity

  $$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega$$

  verifying the compatibility condition

  $$\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n} = 0,$$

  because by divergence theorem (theorem 2.1.3), we have

  $$0 = \int_{\Omega} \nabla\cdot\mathbf{u} = \int_{\partial\Omega} \mathbf{u}\cdot\mathbf{n} = \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}$$

  This boundary condition should be applied when the fluid is confined into a fixed region of space Ω bounded by $\partial\Omega$, where the fluid can not cross the rigid boundaries.

- Neumann boundary condition for normal derivative of velocity field

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \nabla \mathbf{u} \cdot \mathbf{n} = \mathbf{h}$$

or for the Cauchy tensor. For Navier-Stokes problem, this boundary condition can be defined by

$$\mathbf{T} \cdot \mathbf{n} = (-p\mathbf{I} + \nu \nabla \cdot \mathbf{u}) \cdot \mathbf{n} = \mathbf{h}.$$

This boundary condition give us the flux across the boundary.

The type of boundary conditions to apply should depend on the physical conditions. Other type boundary condition is on the boundaries where the inflow is. For the Newtonian case it is enough to define the velocity or the component of surface force over the boundary. The viscoelastic non-Newtonian fluids have memory, this means that the flow of fluid in the domain depends on the deformations which the fluid has been subjected before hold. In this case it is natural to impose boundary conditions on inlet and outlet.

2.6 Stream Function

The stream function is an important analytical tool for the solution of flow problems. Considering two-dimensional plane flow and the components u_1 and u_2 of the velocity vector field $\mathbf{u}$, function of time t and the position (x, y) in the domain Ω, the continuity equations reads

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0.$$

The form of this equation suggests the introduction of a function $\psi \equiv \psi(x, y)$ called the stream function with the property that

$$u_1 = \frac{\partial \psi}{\partial y} \qquad u_2 = -\frac{\partial \psi}{\partial x}$$

or $\nabla \cdot \psi = u_1 - u_2$.

The case of the variable ψ in place of u_1 and u_2 automatically ensures that the continuity equation is satisfied by the Laplace equation in ψ,

$$\Delta \psi = 0,$$

where Δ is the Laplace operator.

The streamline of the flow can be defined by setting the stream function equal to a constant. The streamline is a curve formed by the velocity vectors of each fluid particle at a certain time, i.e., it is the curve where the tangent at each point indicates the direction of flow at that point.

Plotting a family of streamlines, we can understand how fast the fluid is moving at different points, creating a flow visualization. When adjacent streamlines are closer to each other the average fluid velocity is larger. In opposition, when the adjacent streamlines diverge from one another, the average velocity is smaller.

For more details we can read [11].

Chapter 3

Analysis of Navier-Stokes Equations

Navier-Stokes problem can be considered as an auxiliary problem to the Oldroyd-B model, if we consider the viscoelastic extra-stress as body forces (known). In this chapter, we look for the numerical solution of the Navier-Stokes equations for incompressible fluids (constant density). We introduce the mathematical analysis of these equations. Details can be found in [47, 18]. We introduce the variational (weak) formulation of the Navier-Stokes equations and study some classical results of existence and uniqueness of the solutions of these equations with some appropriate boundary conditions. We present some well-known results concerning approximation of Navier-Stokes equations in the context of finite element method and numerical analysis of the approximate problem. For the mathematical theory of finite elements method we refer to [34, 6, 24, 18] and for its numerical implementation we refer to [29, 9]. The final section is dedicated to numerical results obtained on relevant tests.

3.1 The Usual Spaces of Functions

In this section, we introduce some notations, definitions and outline some spaces of functions which are the basis for the modern theory of partial differential equations and which will be useful for our study. Complete presentation on this outline can be found in [14, 1, 34, 52]. We assume throughout this work that Ω is a nonempty, open, bounded domain in Euclidean space $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. In general we will always assume that Ω is simply connected. We will also assume that $\partial\Omega$ is regular enough[1]. Suppose $\mathcal{C}$ be an arbitrary subset of $\mathbb{R}^n$. We denote the closure and interior of $\mathcal{C}$ by $\overline{\mathcal{C}}$ and $\mathring{\mathcal{C}}$ respectively. To indicate vectors we use bold letters and the same letters in normal typface will be used to indicate their components and scalar quantities. We denote the points in Ω or in $\mathbb{R}^d$ by $\mathbf{x} = (x_1, \ldots, x_d)$, a volume element by $d\mathbf{x}(= dx_1, \ldots, dx_d)$ and an element of surface area by ds. If u is a function defined in Ω, we define the support of u, denoted by $supp(u)$, to be the set

$$supp(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We can say $supp(u)$ is the complement of the biggest open set, i.e., the smallest closed set on which u vanishes.

3.1.1 Spaces of continuous functions

We denote the vector space of all continuous functions on $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ by $C^0(\Omega)$ or $C(\Omega)$. We define the partial derivatives of a sufficiently smooth function u of order α by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multi$-$index with $|\alpha| = \sum_{i=1}^{d} \alpha_i$.

Let for any non negative integer m, $C^m(\Omega)$ denotes the vector space of all functions, where all their partial derivatives D^α of orders $0 \leq |\alpha| \leq m$ are continuous on Ω. $C^m(\overline{\Omega})$, $m > 0$ denotes the vector space of all functions in $C^m(\mathcal{O})$ restricted to Ω $(\phi|_\Omega)$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ containing $\overline{\Omega}$. This space is a Banach space with the norm

$$\|u\|_{C^m(\overline{\Omega})} = \max_{0 \leq |\alpha| \leq m} \sup_{\mathbf{x} \in \Omega} |(D^\alpha u)(\mathbf{x})|.$$

[1] The domain Ω is, locally, below the graph of some functions ϕ and the boundary $\partial\Omega$ is represented by the graph of ϕ and the regularity of $\partial\Omega$ is determined by ϕ.

We denote the vector space of all infinitely differentiable functions by $C^\infty(\Omega)$. Actually $C^\infty(\Omega) = \bigcap\limits_{m=0}^{\infty} C^m(\Omega)$. We denote by $C_0^m(\Omega)$ the space of all functions in $C^m(\Omega)$ with compact support.

Let $(V, \|.\|_V)$ and $(W, \|.\|_W)$ be two normed spaces. The set of all linear continuous operators from V into W is denoted by $\mathcal{L}(V; W)$. For $L \in \mathcal{L}(V; W)$ we define the norm

$$\|L\|_{\mathcal{L}(V;W)} = \sup_{\substack{v\in V\\ v\neq 0}} \frac{\|Lv\|_W}{\|v\|_V}.$$

So, $\mathcal{L}(V; W)$ is a normed space. If $W = \mathbb{R}$, the space $\mathcal{L}(V; \mathbb{R})$ is called the dual space of V and is denoted by V'.

3.1.2 The Lebesgue spaces $L^p(\Omega)$

Let Ω be an open, bounded subset of $\mathbb{R}^d$, $d = 2, 3$ with smooth boundary $\partial\Omega$, and consider in Ω the Lebesgue measure. Let $1 \leq p < \infty$. We define the Lebesgue spaces $L^p(\Omega)$, for $1 \leq p < \infty$, as the set of measurable functions u such that $\int_\Omega |u(\mathbf{x})|^p \, d\mathbf{x} < \infty$, i.e.,

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} | \text{ u is measurable and } |u|^p \in L^1(\Omega), \text{ i.e., } \int_\Omega |u(\mathbf{x})|^p \, d\mathbf{x} < \infty \right\}.$$

The functional $\|.\|_{L^p(\Omega)}$ defined by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p}$$

is a norm in $L^p(\Omega)$ provided $1 \leq p < \infty$. $L^p(\Omega)$ is a Banach space with the above norm. The space $L^2(\Omega)$ is, in fact, a Hilbert space with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}$$

For $p = \infty$, $L^\infty(\Omega)$ is a vector spaces of all measurable functions u which are essentially bounded on Ω. In fact, it is a Banach space of essentially bounded real functions with norm

$$\|u\|_{L^\infty} = ess \sup_{\mathbf{x}\in\Omega} |u(\mathbf{x})| = inf \left\{ K \in \mathbb{R} : |u(\mathbf{x})| \leq K, \text{ a.e. } \mathbf{x} \in \Omega \right\}$$

Let $1 \leq p < \infty$. The conjugate exponent q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $u \in L^p(\Omega)$, and $v \in L^q(\Omega)$ with $1 \leq p < \infty$ and q be its conjugate. Then $uv \in L^1(\Omega)$, and we define the Hölder's inequality by

$$\int_\Omega |uv| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

If $p = q = 2$, Hölder's inequality reduces to *Cauchy-Schwarz inequality*

$$|(u, v)| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \qquad \forall u, v \in \Omega \tag{3.1}$$

Another important space is the space of functions in $L^P(\Omega)$ with measure null, i.e.,

$$L_0^p(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega u = 0 \right\}.$$

The space $L_0^p(\Omega)$ is equivalent to the space $L^P(\Omega)/\mathbb{R}$ of functions in $L^p(\Omega)$ defined within a constant.

$L_{loc}^p(\Omega), 1 \leq p \leq \infty$ means the space of measurable functions defined in Ω such that $f \in L^p(K)$ for any compact subset K of Ω. The functions in this space are locally $p-$integrable in Ω in the Lebesgue sense.

If $f \in L_{loc}^p(\Omega)$, then $f \in L_{loc}^1(\Omega)$, i.e., $L_{loc}^p(\Omega) \subset L_{loc}^1(\Omega)$for all $1 \leq \ p \leq \infty$.

3.1.3 The space of distributions

We denote by $\mathcal{D}(\Omega)$ the space of functions in C^∞ with compact support in Ω, and $\mathcal{D}'(\Omega)$ is the space of all linear functionals on $\mathcal{D}(\Omega)$ continuous with respect to its topology [7], i.e., $\mathcal{D}'(\Omega)$ is the dual space of $\mathcal{D}(\Omega)$. This space $\mathcal{D}'(\Omega)$ is called the space of distributions and its elements are called distributions.

$\mathcal{D}(\overline{\Omega})$ is the space of all linear functionals on $\mathcal{D}(\mathcal{O})$ restricted to Ω $(\phi|_\Omega)$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ containing $\overline{\Omega}$.

Corresponding to every $u \in L_{loc}^1(\Omega)$ there is a distribution $F_u \in \mathcal{D}'(\Omega)$ defined by

$$\langle F_u, \phi \rangle = F_u(\phi) = \int_\Omega u(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}, \quad \phi \in \mathcal{D}(\Omega).$$

Here F_u, thus defined, is a linear functional on $\mathcal{D}(\Omega)$. The function u can be identified with the distribution F_u. If $u \in L^2(\Omega)$ we have, for all $\phi \in \mathcal{D}(\Omega)$

$$\langle u, \phi \rangle = \int_\Omega u(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}.$$

If F is a distribution and α is a multi-index, then the derivative $D^\alpha F$ of order α of distribution $F \in \mathcal{D}(\Omega)$ is defined by

$$\langle D^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, D^\alpha \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

We write $\langle F, \phi \rangle$ instead of $F(\phi)$ for the value of F in ϕ for each $\phi \in \mathcal{D}(\Omega)$ and $F \in \mathcal{D}'(\Omega)$.

3.1.4 The Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an arbitrary domain. Sobolev spaces are vector subspaces of various Lebesgue spaces $L^p(\Omega)$. Let $m \geq 0$ be an integer and $1 \leq p \leq \infty$. The Sobolev spaces $W^{m,p}(\Omega)$ are defined to be the set of all functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$. Briefly

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall \alpha, 0 \leq |\alpha| \leq m\}.$$

where $D^\alpha u$ is the distributional partial derivative.

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{if } 1 \leq p < \infty$$

and the corresponding seminorm

$$|u|_{m,p} = \left(\sum_{|\alpha| = m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{if } 1 \leq p < \infty$$

The space $W^{m,\infty}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

and the corresponding seminorm is defined by

$$|u|_{m,\infty} = \max_{|\alpha|=m} \|D^\alpha u\|_{L^\infty(\Omega)} .$$

In particular, when $p = 2$ we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. So we can write

$$H^m(\Omega) = W^{m,2}(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \forall\alpha \text{ such that } 0 \le |\alpha| \le m\right\}$$

Clearly, $W^{0,p}(\Omega) = L^p(\Omega)$ and so we can write $H^0(\Omega) = L^2(\Omega)$.

$H^m(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}$$

In particular, for $m = 1$

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2} \tag{3.2}$$

The seminorm of $H^m(\Omega)$ is defined by:

$$|u|_{H^m(\Omega)} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}$$

Particularly, for $m = 1$ the seminorm is

$$|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)} \tag{3.3}$$

In fact, $H^m(\Omega)$ is Hilbert space with respect to the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v d\mathbf{x}$$

We denote by $H_0^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^m(\Omega)}$. The space $H^{-m}(\Omega)$ is the dual space of $H_0^m(\Omega)$. $H^{-m}(\Omega)$ is equipped with the dual norm

$$\|u\|_{H^{-m}(\Omega)} = \sup_{\substack{v\in H_0^m(\Omega)\\ v\neq 0}} \frac{|\langle u, v\rangle|}{\|v\|_{H^m(\Omega)}}.$$

The space $H^{-m}(\Omega)$ is characterized as the space of derivatives of order up to m of elements of $L^2(\Omega)$.
If Ω has a Lipschitz continuous boundary, $W_0^{m,p}(\Omega)$ is indeed the closure of $C^\infty\left(\overline{\Omega}\right)$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and thus $H^m(\Omega)$ is the closure of $C^\infty\left(\overline{\Omega}\right)$ with respect to the norm $\|\cdot\|_{H^m(\Omega)}$. In other words, $C^\infty\left(\overline{\Omega}\right)$ is dense in $H^m(\Omega)$.

Theorem 3.1.1 *(The Sobolev Embedding Theorem)*
Assume that Ω be a domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$[2] and let $p \in \mathbb{R}$ with $1 \leq p < \infty$. Then the following continuous embeddings hold

- *If $0 \leq sp < d$, then $W^{s,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$, $p* = \frac{dp}{d-sp}$ and there exists $c > 0$ such that $\|v\|_{L^q(\Omega)} \leq c\|v\|_{s,p}$ $\forall v \in W^{s,p}(\Omega), q \in [p, p^*]$*

- *If $sp = d$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, $p \leq q < \infty$, and there exists $c > 0$ such that $\|v\|_{L^q(\Omega)} \leq c\|v\|_{s,p}$ $\forall v \in W^{s,p}(\Omega)$, $p \leq q < \infty$*

- *If $sp > d$, then $W^{s,p}(\Omega) \hookrightarrow C^q(\overline{\Omega})$, and there exists $c > 0$ such that*

$$\|v\|_{C^q(\overline{\Omega})} \leq c\|v\|_{s,p} \ \ \forall v \in W^{s,q}(\Omega),\ q \in \left[0, s - \frac{d}{p}\right]$$

□

Theorem 3.1.2 *(Poincaré's Inequality)*
Assume that Ω be a bounded, connected (in at least one direction), open subset of $\mathbb{R}^d$ $(d = 2, 3)$. Then for all $k > 0$, there exists a constant $C_P = C(d, k, \Omega)$ such that

$$\|u\|_{H^k(\Omega)} \leq C_P\,|u|_{H^k(\Omega)}, \forall u \in H_0^k(\Omega) \tag{3.4}$$

In another form, we can write

$$\|u\|_{L^2(\Omega)} \leq C_P\,\|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega) \tag{3.5}$$

□

[2] A domain $\Omega \in \mathbb{R}^d$ $(d = 2, 3)$ is called a Lipschitz domain if for every $x \in \partial\Omega$, there exists a neighborhood of $\partial\Omega$ which can be represented as the graph of a Lipschitz continuous functions.

Particularly, $\|\nabla u\|_{L^2(\Omega)} = |u|_{H^1(\Omega)}$ is a norm on $H_0^1(\Omega)$ which is equivalent to the norm $\|u\|_{H^1(\Omega)}$. When Ω is bounded subset, the integral $\int_\Omega \nabla u : \nabla v$ is a scalar product over $H_0^1(\Omega)$ which induces norm $\|\nabla u\|_{L^2(\Omega)}$ equivalent to the norm $\|u\|_{H^1(\Omega)}$.

Theorem 3.1.3 *(Green's formula)*
Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$ and $\mathbf{n}$ be a unit outward normal along $\partial\Omega$. Let $u, v \in H^1(\Omega)$, then the integral

$$\int_{\partial\Omega} uvn_i.$$

exists and is finite for each component n_i $(i = 1, \ldots, d)$ of $\mathbf{n}$. Moreover, the formula

$$\int_\Omega \frac{\partial u}{\partial x_i} v = -\int_\Omega u \frac{\partial v}{\partial x_i} + \int_{\partial\Omega} uvn_i \tag{3.6}$$

holds. Let $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have

$$\sum_{i=1}^d \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = -\sum_{i=1}^d \int_\Omega \frac{\partial^2 u}{\partial x_i^2} v + \sum_{i=1}^d \int_{\partial\Omega} n_i \frac{\partial u}{\partial x_i} v \quad \forall u \in H^2(\Omega), \forall v \in H^1(\Omega).$$

i.e.,

$$\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega \Delta u \; v + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \quad \forall u \in H^2(\Omega), \forall v \in H^1(\Omega). \tag{3.7}$$

□

In what follows we will often use spaces of vector functions whose components are in $C^m(\Omega)$, $C^m(\overline{\Omega})$, $L^p(\Omega)$, $H^m(\Omega)$, etc. We will denote the corresponding spaces by a boldface letter i.e., by $\mathbf{C}^m(\Omega)$, $\mathbf{C}^m(\overline{\Omega})$, $\mathbf{L}^p(\Omega)$, $\mathbf{H}^m(\Omega)$, etc.
We denote the space of tensors fields $\mathbf{T} : \Omega \to \mathbb{R}^{d\times d}$ by $[L^p(\Omega)]^{d\times d}$ $(1 \le p < \infty)$ whose components T_{ij} belongs to $L^p(\Omega)$, associated with the norm

$$\|\mathbf{T}\|_{\mathbf{L}^p(\Omega)} = \left(\sum_{i=1}^d \sum_{j=1}^d \|T_{ij}\|_{L^p(\Omega)} \right)^{1/p}.$$

In case of $p = 2$, if $\mathbf{T}, \mathbf{C} \in [L^2(\Omega)]^{d\times d}$, then we define the scalar product by

$$(\mathbf{T}, \mathbf{C})_{\mathbf{L}^2(\Omega)} = \int_\Omega \mathbf{T} : \mathbf{C} d\mathbf{x} = \int_\Omega \sum_{i=1}^d \sum_{j=1}^d T_{ij} C_{ij} d\mathbf{x}.$$

The spaces of symmetric tensor functions whose components belong to $L^2(\Omega)$, $H^m(\Omega)$ $(m \geq 0)$ are denoted by $\mathbf{L}_s^2(\Omega)$, $\mathbf{H}_s^m(\Omega)$, associated with the scalar product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} d\mathbf{x}$$

where $\boldsymbol{\sigma} : \boldsymbol{\tau}$ is the contracted product of two tensors of order n defined by

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^{n} \sigma_{ij}\tau_{ij}.$$

Observe that $\nabla \cdot \boldsymbol{\sigma}$ is a vector whose components are the divergence of line vectors of tensor $\boldsymbol{\sigma}$, i.e.,

$$(\nabla \cdot \boldsymbol{\sigma})_i = \sum_{j=1}^{n} \partial_j \sigma_{ij}$$

(see appendix $(A-9)$).

3.2 Finite Elements Method (FEM)

In this section, we introduce some basic concepts of finite element method (FEM). Details can be found in [6, 34, 19, 37, 3].

FEM is a process for constructing approximate solutions to boundary-value problems. In its simplest form, the finite element method can be interpreted as a Galerkin method, where the solution of continuous problem is approximated by the solution of approach variational problem. Dividing the domain of solution into a finite number of subdomains, the finite elements, the approach variational problem is defined over a finite-dimensional subspace V_h of V (infinite-dimensional function space where the exact solution exists) where h is a discretization parameter. The choice of V_h should be made so that V_h has to be a good approach of V, it means that $\lim\limits_{h\to 0} \dim V_h = +\infty$ and the higher the dimension of V_h better the approximation of the solution of discrete problem to the solution of continuous problem. In fact, the spaces are formed by continuous piecewise polynomials defined over the finite elements with compact support.

For the following, to simplify, we suppose that $\Omega \subset \mathbb{R}^2$ is an open and simply connected. It's possible, without major difficulties, to extend this results to a domain multiple connected in $\mathbb{R}^2$. The three-dimensional case can be found in [19].

The finite element method can be characterized by

(FEM1): Construction of a non-degenerate regular triangulations $\mathcal{T}_h$ over the set $\overline{\Omega}$ of finite elements K, which means

($\mathcal{T}_1$) $\mathcal{T}_h = \bigcup\limits_{K\in\mathcal{T}_h} K = \overline{\Omega}$ and $h = \max\limits_{K\in\mathcal{T}_h} h_K$ is the diameter of $\mathcal{T}_h$ and h_K is the diameter of the circumscribed circle into K;

($\mathcal{T}_2$) For each $K \in \mathcal{T}_h$, the set K is closed and its interior $\mathring{K}$ is non empty;

($\mathcal{T}_3$) The interior of two distinct subset $K_1,\ K_2 \in \mathcal{T}_h$ are disjoint ($\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$) and they have only a common point (vertices) or a common edge or are disjoint;

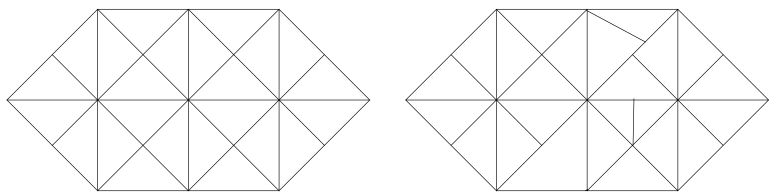


Figure 3.1: *Non-degenerate (admissible) triangulations (on the left) and degenerate (non-admissible) triangulations (on the right).*

($\mathcal{T}_4$) The boundary ∂K is Lipschitz continuous for each $K \in \mathcal{T}_h$;

($\mathcal{T}_4$) There exist positive constants C_1 and C_2 , independent of h, such that

$$C_1 h \leq h_K \leq C_2 \rho_K \ \forall K \in \mathcal{T}_h$$

ρ_K being the diameter of the inscribed circle into K.

(The above condition states that the triangles K of $\mathcal{T}_h$ are approximately the same size.)

(FEM2): Defines a finite set of continuous piecewise polynomial $\{\phi_i, i = 1, \cdots, n\}$ that span a subspace $V_h \subset V$ such that

$$\forall v \in V \quad v = \sum_{i=1}^{n} v_i \phi_i$$

where $v_i = v(x_i, y_i)$ are the degrees of freedom of function v in V_h and $\phi_i(x_j, y_j) = \delta_{ij}$ (δ_{ij} is the Kronecker symbol), with (x_i, y_i) the vertices of $\mathcal{T}_h$, $i, j = 1, \cdots, n$. The functions ϕ_i are called the basis functions of $\mathcal{T}_h$. The set of elements $K \in \mathcal{T}_h$ for which the node $a_i = (x_i, y_i)$ belongs is the $supp(\phi_i), i = 1, \cdots, n$.
Let $\mathbb{P}_k(K)$ be the spaces of polynomials in $\mathbb{R}^2$ of degree $\leq k$. Let $\mathbb{P}_k$ be the polynomial space from $\mathbb{R}^2$ into $\mathbb{R}$ of degree $\leq k$ $(k \geq 0)$ and $\mathbb{P}_k(K)$ the restriction to $K \in \mathcal{T}_h$ of $\mathbb{P}_k$. For $k \geq 1$, we define the space

$$V_h = \left\{ v_h \in C(\overline{\Omega}) : v_{h|_T} \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h \right\} \subset V$$

the discrete space, called finite element space.

(FEM3): Using the variational concepts and the fact that the solution of the continuous problem can be written by

$$v = \sum_{i=1}^{n} v_i \phi_i$$

the finite element method constructs a system of equations, whose solution is solution interpolated of the continuous problem in each vertices.
We introduce triangle $\widehat{K}$ with the vertices $\hat{\mathbf{a}}_1 = (0, 0)$, $\hat{\mathbf{a}}_2 = (1, 0)$ and $\hat{\mathbf{a}}_3 = (0, 1)$ as the reference element. Each arbitrary triangle K with vertices $\mathbf{a}_i = (x_i, y_i)$, $i = 1, 2, 3$, can be obtained as

$$K = F_K(\widehat{K}),$$

where F_K is a suitable invertible affine map (figure 3.2)

$$\begin{aligned} F_K : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ \hat{\mathbf{x}} &\rightarrow B_K \hat{\mathbf{x}} + b_K = \mathbf{x} \end{aligned}$$

where $B_K = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$ is a non-singular matrix and $b_K = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ is a column vector, such that $F_K(\hat{\mathbf{a}}_i) = \mathbf{a}_i, i = 1, 2, 3$.
F_K admits inverse $F_K^{-1}(\mathbf{x}) = B_K^{-1}\mathbf{x} - B_K^{-1} b_K$.
We associate bijectively the function $\hat{v}$ defined over $\widehat{K}$ to all function v defined over

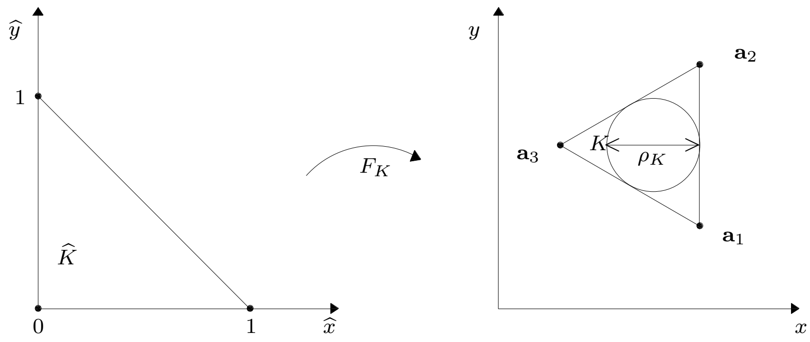


Figure 3.2: *Affine transformation from the reference triangle $\widehat{K}$ to the generic element K in the mesh.*

K, by

$$\forall \hat{\mathbf{x}} \in \widehat{K}, \quad \hat{v}(\hat{\mathbf{x}}) = v(\mathbf{x}) = v\left(F_K\left(\hat{\mathbf{x}}\right)\right) = v \circ F_K\left(\hat{\mathbf{x}}\right).$$

Since F_K is invertible, so we have

$$\forall \mathbf{x} \in K, \quad v(\mathbf{x}) = \hat{v}(\hat{\mathbf{x}}) = \hat{v}\left(F_K^{-1}\left(\mathbf{x}\right)\right) = \hat{v} \circ F_K^{-1}\left(\mathbf{x}\right)$$

Let $|J| > 0$ be the Jacobian of F. We establish a bijective correspondence between a scalar function ϕ defined over K and $\widehat{\phi}$ defined over $\widehat{K}$ by $\widehat{\phi} = \phi \circ F$. The correspondence between a function field $v = (v_1, v_2)$ defined over K and $\widehat{v} = (\widehat{v}_1, \widehat{v}_2)$ defined over $\widehat{K}$ is [9]

$$v_i \circ F = \frac{1}{|J|} \sum_{j=1}^{2} \frac{\partial F_i}{\partial \widehat{x}_i} \widehat{v}_i.$$

This way, we avoid setting n basis functions and is enough to define many basis functions as the number of the degrees of freedom considered in the reference element $\widehat{K}$. So, we can work only over reference triangle $\widehat{K}$.

For the constant $C = C(K)$, we can prove

$$|\hat{v}|_{H^k\left(\widehat{K}\right)} \leq C \left\|B_K\right\|^k |detB_K|^{-1/2} |v|_{H^k(K)}, \quad \forall v \in H^k(K)$$

and

$$|v|_{H^k(K)} \leq C \left\|B_K^{-1}\right\|^k |detB_K|^{1/2} |\hat{v}|_{H^k(K)}, \quad \forall \hat{v} \in H^k(\widehat{K})$$

where $\|.\|$ is the matrix norm associated to the Euclidean norm in $\mathbb{R}^2$. The proof can be found in [34].

We also have the following estimates ([34, 37])

$$\|B_K\| \leq \frac{h_K}{\hat{\rho}}, \qquad\qquad \|B_K^{-1}\| \leq \frac{\hat{h}}{\rho_K},$$

where $\hat{\rho}$ and $\hat{h}$ are the diameters of inscribed and circumscribed circle in $\widehat{K}$. Also,

$$|det(B_K)| = \frac{Area(K)}{Area(\widehat{K})} = 2Area(K) \neq 0.$$

Given a compact subset K of $\mathbb{R}^d$, which is connected and not empty interior, in fact, here K is a triangle. We consider the finite set $\Sigma_K = \{\mathbf{a}_i\}_{i=1}^n$ of distinct points of K and a vectorial space P_K of finite dimension of functions defined over K with real values. We say that Σ_K are P-unisolvent if and only if, for a given n arbitrary real scalars $\alpha_i \quad (i = 1, \ldots, n)$ there exists a function p of the space P_K such that

$$p(\mathbf{a}_i) = \alpha_i \quad (i = 1, \ldots, n) \; \forall \alpha_i \in \mathbb{R}.$$

If the set Σ_K is P-unisolvent, then the triplex (K, P_K, Σ_K) is called Lagrange finite element.

We need to define the following compatibility conditions between two finite elements in order to determine a basis of V_h :

(H1) There is $P_{K_1|_{K'}} = P_{K_2|_{K'}}$ and $\Sigma_{K_1} \cap K' = \Sigma_{K_2} \cap K'$, for all pair $\{K_1, K_2\}$ of adjacent triangles of $\mathcal{T}_h$, with a common side K'=$K_1 \cap K_2$.

(H2) The finite element (K, P_K, Σ_K) is a class of C^0 for all $K \in \mathcal{T}_h$. This means that $P_K \subset C(K)$ and for any side K' of K, the set $\Sigma' = \Sigma_K \cap K'$ is $P'-$unisolvent where $P' = \left\{p_{|_{K'}} : p \in P_K\right\}$

For a finite element (K, P_K, Σ_K), there exists one and only one function $\phi_i \in P_K$ for all $i = 1, \ldots, n$ such that $\phi_i(\mathbf{a}_j) = \delta_{ij}$, for all $j = 1, \ldots, n$ and the only function in P_K which vanishes on Σ_K is the null function. They are the basis functions of V_h. For any function $v \in V_h$, we have

$$v = \sum_{i=1}^{n} v(\mathbf{a}_i)\phi_i.$$

We define the P-interpolation operator of Lagrange for each K such that for all function $v \in C(\overline{\Omega})$, this operator relates the function $\Pi_h^K(v)$ defined by

$$\Pi_h^K(v) = \sum_{i=1}^{n} v(\mathbf{a}_i)\phi_i.$$

So, $\Pi_h^K(v)(\mathbf{a}_j) = \displaystyle\sum_{i=1}^{n} v(\mathbf{a}_i)\phi_i(\mathbf{a}_j) = \sum_{i=1}^{n} v(\mathbf{a}_i)\delta_{ij} = v(\mathbf{a}_j)$.
The interpolant $\Pi_h^K(v)$ is the only function which takes the same values of the given function v at all nodes $\mathbf{a}_i$. We introduce a local interpolation operator with α_i, $1 \leq i \leq n_K$, nodes of K by

$$\Pi_K(v) = \sum_{i=1}^{n_K} v(\alpha_i)\phi_{i_{|K}}, \quad \forall v \in C(K)$$

We can verify that $\Pi_h^K(v)_{|K} = \Pi_K(v_{|K}) \quad \forall K \in \mathcal{T}_h, v \in C(\overline{\Omega})$.

The following theorem gives us an estimate for the interpolation error.

Theorem 3.2.1

Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of regular triangulations of $\overline{\Omega}$ whose elements verifies (**H1**) *and* (**H2**). *Let $k \geq 1$ be an integer. For $m \in \{0, 1\}$, there is a constant C, independent of h, such that*

$$|v - \Pi_K(v)|_{H^m(\Omega)} \leq Ch^{k+1-m} |v|_{H^{k+1}(\Omega)} \qquad \forall v \in H^{k+1}(\Omega). \tag{3.8}$$

Moreover,

$$\inf_{v_h \in V_h} |v - v_h|_{H^m(\Omega)} \leq Ch^{k+1-m} |v|_{H^{k+1}(\Omega)} \qquad \forall v \in H^{k+1}(\Omega). \tag{3.9}$$

□

The proof can be found in [34, 37]

3.3 Mathematical Analysis for Steady Navier-Stokes Problem

The steady Navier-Stokes equations of a homogeneous incompressible Newtonian fluid, which describe the flow motion, independent of time, read as

$$\begin{cases} (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = \mathbf{f} \\ \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{3.10}$$

where $\mathbf{u}$ is the velocity field of the fluid, $\mathbf{f}$ is a given external force field per unit mass, p is the ratio between the pressure and the density which is known as kinematic pressure ($[p] = m^2/s^2$), $\nu > 0$ is the ratio between its dynamic viscosity and density known as constant kinematic viscosity ($[\nu] = m^2/s$). Here Ω is a bounded domain of $\mathbb{R}^d(d{=}2, 3)$ with Lipschitz continuous boundary $\partial\Omega$. To close mathematical formulation and obtain a well-posed problem, the above equations need to be supplemented by some boundary conditions. For simplicity, we consider the case in which the system of differential equations (3.10) is equipped with the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega \qquad \text{(adherence conditions).} \tag{3.11}$$

Using the divergence theorem, we have

$$0 = \int_\Omega \nabla \cdot \mathbf{u} = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}.$$

So, for the incompressible fluids, the Dirichlet boundary condition g satisfies the compatibility condition

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0 \tag{3.12}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$.

The condition $\mathbf{g} = 0$ is called the homogeneous Dirichlet boundary conditions (or no slip boundary conditions), which describes a fluid confined into a domain Ω with fixed boundary (the boundary is at rest).

For simplicity, we take $\mathbf{g} = 0$. The extension to nonhomogeneous Dirichlet boundary conditions on the sufficiently regular data being straight-forward. So, with the homogeneous Dirichlet boundary conditions defined over Ω, we can write the steady Navier-Stokes problem as follows:

Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$, find $(\mathbf{u}, p)$ such that

$$\begin{cases} (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = \mathbf{f}, & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \\ \mathbf{u} = 0, & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

If the velocity of the flow is small enough, then the nonlinear convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is negligible. So, for slow viscous flows, from the Navier-Stokes problem (3.13), we obtain the following Stokes' problem:

Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$, find $(\mathbf{u}, p)$ such that

$$\begin{cases} \nabla p - \nu \Delta \mathbf{u} = \mathbf{f}, & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \\ \mathbf{u} = 0, & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

3.3.1 Variational formulation of Navier-Stokes problem

The variational formulation (weak formulation) of the Navier-Stokes equations consists of integral equations over Ω which is obtained by taking integral over the domain of the scalar product of the momentum equation and the continuity equation with appropriate test functions, and applying the Green integration formula. Following Ladyzhenskaya (1959), we assume that $\mathbf{u} \in \mathbf{C}^2(\Omega) \cap \mathbf{C}^0\left(\overline{\Omega}\right)$ and $p \in C^1(\Omega) \cap C^0\left(\overline{\Omega}\right)$ are the classical (or strong) solution of (3.13) and $\mathbf{f} \in \mathbf{C}(\Omega)$. Consider two Hilbert spaces $\mathbf{V} = \mathbf{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$ and let $\mathbf{v} \in \mathbf{V}$ and $q \in Q$ be two arbitrary test functions. Taking the scalar product between the momentum equation and $\mathbf{v}$, we obtain

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} + \int_\Omega \nabla p \cdot \mathbf{v} - \nu \int_\Omega \Delta \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \tag{3.15}$$

By using the Green's formulas (theorem 3.1.3) to (3.15) and taking into account that $\mathbf{v}$ vanishes on the boundary, the variational form of the momentum equation is

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \tag{3.16}$$

Multiplying $\nabla \cdot \mathbf{u} = 0$ by q and integrating over Ω, we obtain

$$\int_\Omega \nabla \cdot \mathbf{u}\, q = -\int_\Omega q \nabla \cdot \mathbf{u} = 0 \tag{3.17}$$

The variational formulation of the Navier-Stokes problem (3.13) reads:

Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} \displaystyle\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} + \nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}, & \forall \mathbf{v} \in \mathbf{V} \\ \displaystyle\int_\Omega \nabla\cdot\mathbf{u}\, q = 0, & \forall q \in Q \end{cases} \tag{3.18}$$

Lemma 3.3.1

Problem (3.13) *and Problem* (3.18) *are equivalent.* □

Proof

It is immediate that a smooth solution $(\mathbf{u}, p)$ of (3.13) is the solution of (3.18) , i.e., $(\mathbf{u}, p)$ is also a weak solution of the Navier-Stokes problem.

Conversely, assuming $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ belongs to $\mathbf{C}^2\left(\overline{\Omega}\right) \times C^1\left(\overline{\Omega}\right)$ where Ω is class C^1, is a solution of (3.18), choosing a test function $\mathbf{v} \in \mathcal{D}(\Omega)$, and by applying Green's formula we obtain

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} + \nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} \Leftrightarrow$$

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega \nabla p\cdot\mathbf{v} - \int_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n} - \nu\int_\Omega \Delta\mathbf{u}\mathbf{v} + \nu\int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}$$

Since $\mathbf{v} \in \mathcal{D}(\Omega)$, the *supp*$(\mathbf{v})$ is compact and we have

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega \nabla p\cdot\mathbf{v} - \nu\int_\Omega \Delta\mathbf{u}\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v},\ \forall \mathbf{v} \in \mathcal{D}(\Omega).$$

By density,

$$\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p - \nu\Delta\mathbf{u} - \mathbf{f}] : \mathbf{v} = 0,\ \forall \mathbf{v} \in \mathbf{L^2}(\Omega) \Leftrightarrow$$

$(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p - \nu\Delta\mathbf{u} - \mathbf{f} = 0$ a.e. in Ω

In fact, $(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p - \nu\Delta\mathbf{u} - \mathbf{f} = 0$ in Ω, since $(\mathbf{u}, p) \in \mathbf{C}^2(\Omega) \cap \mathbf{C}^0\left(\overline{\Omega}\right) \times C^1(\Omega) \cap C^0\left(\overline{\Omega}\right)$. Since $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, so $\mathbf{u} = 0$ on $\partial\Omega$. In this way we can conclude that the solution of (3.18) is also a solution (weak) of Problem (3.13). □

3.3.2 Abstract formulation

We introduce the variational formulation of the previous problem in a general abstract formulation that is suitable for many elliptic problems. Let us introduce continuous bilinear forms $a(.,.) : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}$ and $b(.,.) : \mathbf{V} \times Q \longrightarrow \mathbb{R}$ as

$$a(\mathbf{u}, \mathbf{v}) = \nu\,(\nabla \mathbf{u}, \nabla \mathbf{v}) = \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \tag{3.19}$$

$$b(\mathbf{v}, p) = -\int_\Omega p \nabla \cdot \mathbf{v} \tag{3.20}$$

Besides, we define $c(.;.,.) : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}$ as the trilinear form associated with the nonlinear convective term by

$$c(\mathbf{w}; \mathbf{u}, \mathbf{v}) = ((\mathbf{w} \cdot \nabla)\,\mathbf{u}, \mathbf{v}) = \int_\Omega [(\mathbf{w} \cdot \nabla)\mathbf{u}] \cdot \mathbf{v} = \int_\Omega \sum_{i,j=1}^{d} w_j \frac{\partial u_i}{\partial x_j} v_i \tag{3.21}$$

Lemma 3.3.2

The forms $a : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}$, $b : \mathbf{V} \times Q \longrightarrow \mathbb{R}$, *and* $c : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}$ *defined by* (3.19), (3.20) *and* (3.21) *respectively, are continuous with respect to their arguments. Moreover,* $a(\cdot,\cdot)$ *is coercive (*$\mathbf{V}$*-elliptic), i.e.,*

$$\exists \alpha > 0 : \ a(\mathbf{v}, \mathbf{v}) \geq \alpha \left\| \mathbf{v} \right\|^2_{\mathbf{H}^1(\Omega)}, \quad \forall \mathbf{v} \in \mathbf{V}.$$

□

Proof

The continuity of bilinear forms $a(\cdot,\cdot)$ is an immediate consequence of the Cauchy-Schwarz inequality (Hölder inequality with $p = q = 2$). In fact, $\forall\ \mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $\forall\ q \in Q$

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| = \left| \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \right| &\underbrace{\leq}_{Holder} \nu \left\| \nabla \mathbf{u} \right\|_{\mathbf{L}^2(\Omega)} \left\| \nabla \mathbf{v} \right\|_{\mathbf{L}^2(\Omega)} \\ &\underbrace{\leq}_{def.norm\ \mathbf{H}^1} \nu \left\| \mathbf{u} \right\|_{\mathbf{H}^1(\Omega)} \left\| \mathbf{v} \right\|_{\mathbf{H}^1(\Omega)} \end{aligned} \tag{3.22}$$

and

$$|b(\mathbf{u}, q)| = \left| \int_\Omega q \nabla \cdot \mathbf{u} \right| \underbrace{\leq}_{Holder} \left\| \nabla \cdot \mathbf{u} \right\|_{\mathbf{L}^2(\Omega)} \left\| q \right\|_{\mathbf{L}^2(\Omega)} \underbrace{\leq}_{def.norm\ \mathbf{H}^1} \left\| \mathbf{u} \right\|_{\mathbf{H}^1(\Omega)} \left\| q \right\|_{\mathbf{L}^2(\Omega)}. \tag{3.23}$$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ be arbitrary functions. Thanks to the Sobolev embedding theorem (theorem 3.1.1), we have $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ for $d = 2, 3$ and consequently $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$. Then $\mathbf{wv} \in \mathbf{L}^2(\Omega)$.

Considering the expression $c(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}$ componentwise, we have

$$\begin{aligned}
\left| \int_\Omega w_i \frac{\partial u_k}{\partial x_i} v_k \right| &\leq \|w_i v_k\|_{L^2(\Omega)} \left\| \frac{\partial u_k}{\partial x_i} \right\|_{L^2(\Omega)} \\
&= \left[\int_\Omega (w_i v_k)^2 \right]^{\frac{1}{2}} \left[\int_\Omega \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right]^{\frac{1}{2}} \\
&\leq \left(\int_\Omega w_i^4 \right)^{\frac{1}{4}} \left(\int_\Omega v_k^4 \right)^{\frac{1}{4}} \left[\int_\Omega \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right]^{\frac{1}{2}} \\
&\leq \|w_i\|_{L^4(\Omega)} \|v_k\|_{L^4(\Omega)} \|u_k\|_{H^1(\Omega)}
\end{aligned}$$

Using the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, there is a positive constant C such that

$$\left| \int_\Omega w_i \frac{\partial u_k}{\partial x_i} v_k \right| \leq C^2 \|w_i\|_{H^1(\Omega)} \|v_k\|_{H^1(\Omega)} \|u_k\|_{H_0^1(\Omega)} \leq C^2 \|w_i\|_{H^1(\Omega)} \|v_k\|_{H^1(\Omega)} \|u_k\|_{H^1(\Omega)}$$

Therefore, owing to the Poincaré inequality, we can conclude that, for positive constant C_1 we have

$$|c(\mathbf{w}; \mathbf{u}, \mathbf{v})| \leq C_1 \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega). \tag{3.24}$$

So, $c(.; ., .)$ is continuous on $\mathbf{H}_0^1(\Omega)$.

The coercivity of bilinear form $a(\cdot, \cdot)$ is an immediate consequence of the norms and Poincaré inequality (3.1.2),

$$a(\mathbf{u}, \mathbf{u}) = \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{u} = \nu \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} = \nu \, |\nabla \mathbf{u}|_{\mathbf{H}^1(\Omega)} \geq \frac{\nu}{C_P} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}$$

with C_P the constants of Poincaré inequality. ■

Lemma 3.3.3

Let $\mathbf{w} \in \mathbf{H}^1(\Omega)$ with $\nabla \cdot \mathbf{w} = 0$ in Ω and $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and let $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$. Then, we have

$$c(\mathbf{w}; \mathbf{v}, \mathbf{v}) = 0. \tag{3.25}$$

and

$$c(\mathbf{w};\mathbf{u},\mathbf{v}) = -c(\mathbf{w};\mathbf{v},\mathbf{u}). \tag{3.26}$$

□

Proof

The properties (3.25) and (3.26) are equivalent. Because,

$$\begin{aligned} c(\mathbf{w};\mathbf{u},\mathbf{v}) = -c(\mathbf{w};\mathbf{v},\mathbf{u}) \quad &\Leftrightarrow \quad c(\mathbf{w};\mathbf{u},\mathbf{v}) + c(\mathbf{w};\mathbf{v},\mathbf{u}) = 0 \\ &\Leftrightarrow \quad c(\mathbf{w};\mathbf{u}+\mathbf{v},\mathbf{v}+\mathbf{u}) = 0 \Leftrightarrow c(\mathbf{w},\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}) = 0. \end{aligned}$$

So, it is sufficient to show that $c(\mathbf{w};\mathbf{v},\mathbf{v}) = 0$.

Suppose $\mathbf{u},\mathbf{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and $\mathbf{w} \in \mathbf{H}^1(\Omega)$. We have

$$\begin{aligned} c(\mathbf{w};\mathbf{v},\mathbf{v}) &= \sum_{i,j=1}^{d} \int_\Omega w_j \frac{\partial v_i}{\partial x_j} v_i = \sum_{i,j=1}^{d} \int_\Omega w_j \frac{1}{2}\frac{\partial (v_i^2)}{\partial x_j} \\ &= -\frac{1}{2}\sum_{i,j=1}^{d} \left(\int_\Omega \frac{\partial w_j}{\partial x_j} v_i^2 + \int_{\partial\Omega} w_j n_j v_i^2 \right), \quad \text{by Green's formula} \end{aligned}$$

Applying the hypothesis $\nabla \cdot \mathbf{w} = 0$ and $\mathbf{w}\cdot\mathbf{n} = 0$ on $\partial\Omega$, we have

$$c(\mathbf{w};\mathbf{v},\mathbf{v}) = 0 \ \forall \mathbf{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}).$$

By density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ into $\mathbf{H}^1(\Omega)$, we have

$$c(\mathbf{w};\mathbf{v},\mathbf{v}) = 0, \ \forall \mathbf{v} \in \mathbf{H}^1(\Omega).$$

■

So, we can reformulate the variational formulation of Navier-Stokes problem as follows:

Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, find $\mathbf{u} \in \mathbf{V},\ p \in Q$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) = 0 & \forall q \in Q \end{cases} \tag{3.27}$$

Note that, for any fixed $\mathbf{w} \in \mathbf{V} = \mathbf{H}_0^1(\Omega)$, the map $\mathbf{v} \longrightarrow c(\mathbf{w};\mathbf{w},\mathbf{v})$ is linear and continuous on $\mathbf{V}$, i.e., it is an element of $\mathbf{V}' = \mathbf{H}_0^{-1}(\Omega)$. Let us define $\mathbf{V}_{div}$, the subspace of $\mathbf{V} = \mathbf{H}_0^1(\Omega)$ of divergence free functions as $\mathbf{V}_{div} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) :\ \nabla\cdot\mathbf{v} = 0\}$.

$\mathbf{v} \in \mathbf{V}_{div}$ implies the bilinear form $b(\mathbf{v}, p) = 0$. We can write the alternative weak formulation of the Navier-Stokes problem (3.27) as

Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$, find $\mathbf{u} \in \mathbf{V}_{div}$ such that

$$a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_{div} \tag{3.28}$$

Remark

If $(\mathbf{u}, p)$ is a solution of (3.27), then $\mathbf{u}$ is a solution of (3.28). The converse is also true according to the following results:

Lemma 3.3.4

If if $\mathbf{u} \in \mathbf{V}_{div}$ is a solution of the problem (3.28)*, then there exists a unique $p \in Q$ such that $(\mathbf{u}, p)$ is a solution of the problem* (3.27)*.* □

The proof can be found in [35, 34].

3.3.3 Existence and uniqueness of the solution

In this section we present two theorems: one is uniqueness and the other is existence of a solution of the two-dimensional steady Navier-Stokes problem. The solution of the Navier-Stokes problem (3.27) is generally known as non-unique. Only when the data is small enough and the viscosity is high enough, we achieved uniqueness. At first define the space

$$H_{div} := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}, \tag{3.29}$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$. The space H_{div} is equipped with $\|\nabla \cdot\|_{L^2(\Omega)}$.

Theorem 3.3.1

Let $\mathbf{f} \in \mathbf{H}_{div}$ with

$$\frac{1}{\nu^2}\|\mathbf{f}\|_{L^2(\Omega)} < \frac{1}{C_1\sqrt{C_P}} \Leftrightarrow \frac{C_1\sqrt{C_P}}{\nu^2}\|\mathbf{f}\|_{L^2(\Omega)} < 1$$

where $C_1 > 0$ is the constant appearing in (3.24)*and C_P is the constant of Poincaré inequality. Then there exists a unique solution $\mathbf{u} \in \mathbf{V}_{div}$ to the problem* (3.28)*.* □

The proof can be found in [34, 18].

Let us consider the general case of a nonhomogeneous Dirichlet boundary condition $\mathbf{u} = \mathbf{g}$.

The variational form of Navier-Stokes problem consists in

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \text{ such that} & \\ a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \text{ in } \Omega & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) = 0 & \forall q \in Q \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \end{cases} \tag{3.30}$$

Theorem 3.3.2

Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$ satisfying $\int_\Omega \mathbf{g} \cdot \mathbf{n} ds = 0$, there exists at least one pair $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ solution of (3.30) or equivalently solution of (3.10) – (3.13).

$\square$

Proof can be found in [16, 34, 18].

3.4 Stability

The stability criterion is essential for physical problems. A mathematical problem is usually considered physically realistic if a small change in given data produces correspondingly a small change in the solution, i.e., the solution depends continuously on data. The following lemma guarantees that the mathematical problem is physically realistic.

Lemma 3.4.1

Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $p \in \mathbf{L}_0^2(\Omega)$ be the solution of (3.30) and the hypothesis of theorem 3.3.2 holds. Then the following energy inequality holds:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \frac{\nu}{C_P} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \tag{3.31}$$

$\square$

Proof

Multiplying the first of (3.10) by $\mathbf{u}$ and integrating over the domain Ω, we obtain

$$-\nu \int_\Omega \Delta \mathbf{u} \cdot \mathbf{u} + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} + \int_\Omega \nabla p \cdot \mathbf{u} = \int_\Omega \mathbf{f} \cdot \mathbf{u} \tag{3.32}$$

But

$$\begin{aligned}
\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} &= \sum_{i,j=1}^{d} \int_\Omega u_j \frac{\partial u_i}{\partial x_j} u_i = \sum_{i,j=1}^{d} \int_\Omega u_j \frac{1}{2} \frac{\partial (u_i^2)}{\partial x_j} \\
&= -\frac{1}{2} \sum_{i,j=1}^{d} \left(\int_\Omega \frac{\partial u_j}{\partial x_j} u_i^2 + \int_{\partial\Omega} u_j n_j u_i^2 \right) = 0
\end{aligned} \tag{3.33}$$

and

$$\int_\Omega \nabla p \cdot \mathbf{u} = -\int_\Omega p \nabla \cdot \mathbf{u} + \int_{\partial\Omega} p \mathbf{n} \cdot \mathbf{u} = 0 \tag{3.34}$$

Moreover, by the Poincaré inequality

$$\begin{aligned}
\frac{\nu}{C_P} \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \leq \nu \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} = \nu \int_\Omega |\nabla \mathbf{u}|^2 &= \nu \int_\Omega |\nabla \mathbf{u}|^2 - \int_{\partial\Omega} \nabla \mathbf{u} \cdot \mathbf{u} \cdot \mathbf{n} \\
&= -\nu \int_\Omega \Delta \mathbf{u} \cdot \mathbf{u}
\end{aligned} \tag{3.35}$$

and by the Hölder's inequality

$$\int_\Omega \mathbf{f} \cdot \mathbf{u} \leq \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}$$

So, by (3.33), (3.34) and (3.35)

$$\begin{aligned}
\frac{C_P}{\nu} \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} &\leq -\nu \int_\Omega \Delta \mathbf{u} \cdot \mathbf{u} + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} + \int_\Omega \nabla p \cdot \mathbf{u} = \int_\Omega \mathbf{f} \cdot \mathbf{u} \\
&\leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \Leftrightarrow \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \frac{\nu}{C_P} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}
\end{aligned}$$

■

3.5 Numerical Analysis for the Navier-Stokes problem

Let $\mathcal{T}_h$ be a non-degenerate triangulations of Ω, with $h > 0$ the discretization parameter of the mesh. Let $\mathbf{V}_h$ and Q_h be two finite-dimensional spaces for the velocity field and the pressure, respectively, such that $\mathbf{V}_h \subset \mathbf{H}^1(\Omega)$ and $Q_h \subset L^2_0(\Omega)$. We set

$$\mathbf{V}_h^0 = \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega) \quad \text{and} \quad M_h = Q_h \cap L_0^2(\Omega).$$

In these spaces, the problem (3.27) is approximated by

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times M_h \ \text{ such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v_h} \in \mathbf{V}_h^0 \\ b(\mathbf{u_h}, q_h) = 0 \quad \forall q_h \in M_h \end{cases} \tag{3.36}$$

The existence and uniqueness of the problem (3.36) is generated by the fact that the discrete space $\mathbf{V}_h^0$ and M_h verify a compatibility condition known as '*consistency* condition', '*inf-sup* condition' or LBB-condition, which reads as follows:

There exists $\beta > 0$ (independent of h) such that

$$\inf_{q_h \in M_h \setminus \{0\}} \sup_{\mathbf{v}_h \in \mathbf{V}_h^0 \setminus \{0\}} \frac{|(q_h, \nabla \cdot \mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\mathbf{V}_h^0} \|q_h\|_{M_h}} \geqslant \beta. \tag{3.37}$$

The property (3.37) is necessary for the well-posedness of the discrete problem.

Lemma 3.5.1

Problem (3.36) *has a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times M_h$.

Furthermore, $(\mathbf{u}_h, p_h)$ *converges to solution* $(\mathbf{u}, p)$ *of the problem* (3.27) *, that is*

$$\lim_{h \to 0} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} + \lim_{h \to 0} \|p - p_h\|_{L^2(\Omega)} = 0 \tag{3.38}$$

□

We need to choose the discrete spaces of the velocity and pressure very carefully. In fact, 'spurious oscillations' phenomena for the unknown pressure may appear if we do not ensure some kind of compatibility between the spaces involved in the approximation. The discrete LBB-condition allows to obtain the correct setting of discrete spaces.

Let $\mathcal{T}_h,\ h > 0$ be a non-degenerate uniformly regular triangulation defined over a polygonal domain $\Omega \subset \mathbb{R}^2$ such that $\overline{\Omega} = \underset{K \in \mathcal{T}_h}{\cup} K$. Consider the following pair of spaces $(\mathbf{V}_h^0, M_h)$:

$$\mathbf{V}_h^0 = \big\{ \mathbf{v}_h \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{H}_0^1(\Omega) \mid v_{h|K} \in \mathbb{P}_2(K), \forall K \in \mathcal{T}_h \big\}, \tag{3.39}$$

$$M_h = \left\{ q_h \in C(\overline{\Omega}) \cap L_0^2(\Omega) \mid q_{h|K} \in \mathbb{P}_1(K),\, \forall K \in \mathcal{T}_h \right\} \subset Q. \tag{3.40}$$

corresponding to the Hood-Taylor finite element method. In this specific case, the corresponding LBB-condition is satisfied.
We have the following results, which is classical and can be found in [6].

Theorem 3.5.1
Suppose that $\mathcal{T}_h$ is non-degenerate and has no triangle with two edges on $\partial\Omega$. Let $\mathbf{V}_0^h$ and M_h be respectively as in (3.39) *and* (3.40). *Then the LBB-condition* (3.37) *is satisfied.* □

The next theorem deals with the error estimate for the Navier-Stokes approximation of (3.36) using the Hood-Taylor finite element method. Proof can be found in [18].

Theorem 3.5.2
Let the solution $(\mathbf{u}, \mathbf{p})$ *of the Navier-Stokes system* (3.13) *satisfy*

$$\mathbf{u} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega),\ p \in H^k(\Omega) \cap L_0^2(\Omega),\ k = 1, 2.$$

If the triangulation $\mathcal{T}_h$ is regular and it has no triangle with two edges on $\partial\Omega$, then the solution $(\mathbf{u}_h, p_h)$ *of the problem* (3.36) *with $\mathbf{V}_h^0$ and M_h give by* (3.39) *and* (3.40) *satisfies the following error estimates:*

$$|\mathbf{u} - \mathbf{u}_h|_{\mathbf{H}^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \le C_1 h^k \left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)} + |p|_{H^k(\Omega)} \right), \quad k = 1, 2. \tag{3.41}$$

□

We discuss about the numerical stability. The trilinear form $c(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{u} \cdot \nabla)\, \mathbf{u} \cdot \mathbf{v}$ does not contribute to energy system at differential level in the Navier-Stokes equations as we see from section 3.4. When $\mathbf{v} = \mathbf{u}$, we have

$$\int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} = \frac{1}{2} \int_\Omega \mathbf{u} \cdot \nabla |\mathbf{u}|^2 = -\frac{1}{2} \int_\Omega \nabla \cdot \mathbf{u}\, |\mathbf{u}|^2 + \frac{1}{2} \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}\, |\mathbf{u}|^2 = -\frac{1}{2} \int_\Omega \nabla \cdot \mathbf{u}\, |\mathbf{u}|^2 \tag{3.42}$$

Since $\nabla \cdot \mathbf{u} = 0$, so $-\frac{1}{2}\int_\Omega \nabla \cdot \mathbf{u}\,|\mathbf{u}|^2 = 0$. But in case of discretized problem (3.36), the term $-\frac{1}{2}\int_\Omega \nabla \cdot \mathbf{u}\,|\mathbf{u}|^2$ can not be zero. To overcome this problem, we can add an additional term

$$\frac{1}{2}\left((\nabla \cdot \mathbf{u}_h)\mathbf{u}_h, \mathbf{v}_h\right) = \frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{u}_h)\, \mathbf{u}_h \cdot \mathbf{v}_h. \tag{3.43}$$

to the equation $(3.36)_1$ to make it consistent. Since for the incompressibility condition $\nabla\cdot\mathbf{u} = 0$, the additional term (3.43) reduces to zero and the modification is consistent. So, the modified approximate problem can be written as follows:

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times M_h \quad \text{such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \dfrac{1}{2}\left((\nabla \cdot \mathbf{u}_h)\mathbf{u}_h, \mathbf{v}_h\right) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \qquad \forall \mathbf{v_h} \in \mathbf{V}_h^0 \\ b(\mathbf{u_h}, q_h) = 0 \qquad \forall q_h \in M_h \end{cases} \tag{3.44}$$

We can prove that the problem (3.44) is stable.

Lemma 3.5.2

If $\mathbf{u}_h \in \mathbf{V}_h^0$ is a solution of (3.44),then the following energy inequality holds:

$$\|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \le \frac{\nu}{C_P}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \tag{3.45}$$

□

Proof

Putting $\mathbf{v}_h = \mathbf{u}_h$ in $(3.44)_1$.

Now, we have

$$\begin{aligned} c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h) + \frac{1}{2}\left((\nabla \cdot \mathbf{u}_h)\, \mathbf{u}_h, \mathbf{u}_h\right) = 0 \;\; \text{by using (3.43)} \\ \Rightarrow \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{u}_h + \frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{u}_h)\mathbf{u}_h \cdot \mathbf{u}_h = 0 \end{aligned}$$

So, with all the procedure applied to proof the lemma (3.4.1), we can obtain (3.45).

■

We can say that the inequality (3.45) gives the estimate of stability. In fact, the discrete part of (3.31) is the inequality (3.45).

3.5.1 Algebraic form of the approach Navier-Stokes problem

Let the following finite element spaces $\mathbf{V}_h^0$ and M_h are given by (3.39) and (3.40) such that

- $dim(V_h^0) = N$,where N is the total number of vertices and the midpoints of the edges of the triangular meshes which are interior of Ω (excluding the boundary $\partial\Omega$, since the velocity over $\partial\Omega$ is known in case of Dirichlet problem).
- $dim(M_h) = M$, where M is the total number of vertices of mesh's triangles.

Let $\mathbf{V}_h^0 = V_h^0 \times V_h^0$. We recall that the pair of spaces $(\mathbf{V}_h^0, M_h)$ corresponds to the *Hood-Taylor* finite element method $\mathbb{P}_2 - \mathbb{P}_1$, and satisfies a compatibility condition discrete *LBB*.

We want to solve the approximate problem (3.36) which can be written as

Find $(\mathbf{u_h}, p_h) \in \mathbf{V}_h^0 \times M_h$ i.e., $\left(u_1^h, u_2^h, p_h\right) \in V_h^0 \times V_h^0 \times M_h$ such that

$$\begin{cases} \nu \displaystyle\int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v}_h + \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h + \frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{u}_h)\, \mathbf{u}_h \cdot \mathbf{v}_h \\ -\displaystyle\int_\Omega p_h \nabla \cdot \mathbf{v}_h = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h, \;\; \forall \mathbf{v}_h = \left(v_1^h, v_2^h\right) \in \mathbf{V}_h^0 \\ \displaystyle\int_\Omega \nabla \cdot \mathbf{u}_h \, q_h = 0, \;\; \forall q_h \in M_h \end{cases} \tag{3.46}$$

Since the dimension of V_h^0 and M_h are N and M respectively, let V_h^0 has a Lagrange basis $\{\varphi_i\}_{i=1,\cdots,N}$ and M_h has a Lagrange basis $\{\psi_i\}_{i=1,\cdots,M}$. Let us write the approximate solutions $\mathbf{u_h} = (u_{1,h}, u_{2,h})$ and p_h in the basis of V_h^0 and M_h as

$$\mathbf{u_h} = (u_{1,h}, u_{2,h}) = \left(\sum_{j=1}^{N}(u_1)_j\varphi_j, \sum_{j=1}^{N}(u_2)_j\varphi_j\right), \quad p_h = \sum_{l=1}^{M} p_l\varphi_l$$

Let $(\mathbf{v}_h, q_h) = (v_{1,h}, v_{2,h}, q_h) = (\varphi_i, \varphi_i, \psi_k)$ be the test functions $\varphi \in V_h^0$ and $\psi \in M_h$. Setting $\mathbf{v_h} = (v_{1,h}, v_{2,h}) = (\{\varphi_i\}, 0)$ and $(0, \{\varphi_i\})$, for $i = 1, \cdots, N$ and $q_h = \{\psi_l\}_{l=1,\cdots,M}$, we obtain an equivalent coupled set of scalar equations

$$\begin{cases} \displaystyle\int_\Omega \left[\nu\nabla u_{1,h}\cdot\nabla\varphi_i + (\mathbf{u}_h\cdot\nabla)u_{1,h}\varphi_i + \frac{1}{2}(\nabla\cdot\mathbf{u}_h)u_{1,h}\varphi_i - p_h\frac{\partial\varphi_i}{\partial x} - f_1\varphi_i\right] = 0,\ i = 1,\cdots,N \\ \displaystyle\int_\Omega \left[\nu\nabla u_{2,h}\cdot\nabla\varphi_i + (\mathbf{u}_h\cdot\nabla)u_{2,h}\varphi_i + \frac{1}{2}(\nabla\cdot\mathbf{u}_h)u_{2,h}\varphi_i - p_h\frac{\partial\varphi_i}{\partial y} - f_2\varphi_i\right] = 0,\ i = 1,\cdots,N \\ \displaystyle\int_\Omega (\nabla\cdot\mathbf{u}_h)\,\psi_l = 0,\ l = 1,\cdots,M \end{cases} \tag{3.47}$$

The above system of equations can be written as a non-symmetric matricial equation:

$$\begin{bmatrix} \nu\mathbf{A} & 0 & \mathbf{B}_x \\ 0 & \nu\mathbf{A} & \mathbf{B}_y \\ \mathbf{B}_x^t & \mathbf{B}_y^t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} c(\mathbf{u}_1) \\ c(\mathbf{u}_2) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ 0 \end{bmatrix} \tag{3.48}$$

where $\mathbf{u}_i^t = [u_i^1 \cdots u_i^N]$, $i = 1, 2$, for N nodal velocities, $\mathbf{p}^t = [p_1 \cdots p_M]$, for M nodal pressure and

$$\mathbf{A} = [\mathbf{A}_{ij}]_{N\times N} = \int_\Omega \frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}, i, j = 1, \cdots, N.$$

$$\mathbf{B}_x = [\mathbf{B}_{xil}]_{N\times M} = \left[\int_\Omega \frac{\partial\varphi_i}{\partial x}\psi_l\right]_{N\times M}$$

$$\mathbf{B}_y = [\mathbf{B}_{yil}]_{N\times M} = \left[\int_\Omega \frac{\partial\varphi_i}{\partial y}\psi_l\right]_{N\times M}, \text{ for } i = 1, \cdots, N, j = 1, \cdots, M$$

$$\mathbf{F}_x = \left[\int_\Omega f_1\varphi_i\right]_{N\times 1}, \text{ and } \mathbf{F}_y = \left[\int_\Omega f_2\varphi_i\right]_{N\times 1}.$$

The nonlinear $N \times 1$ vector $\mathbf{c}(\mathbf{u}_i)$ is given as

$$c_i(\mathbf{u}_l) = \int_\Omega \left[\left(\sum_{j=1}^{N} u_{1,j}\varphi_j \right) \left(\sum_{k=1}^{N} (u_l)_k \frac{\partial \varphi_k}{\partial x} \right) + \left(\sum_{j=1}^{N} u_{2,j}\varphi_j \right) \left(\sum_{k=1}^{N} (u_l)_k \frac{\partial \varphi_k}{\partial y} \right) \right]$$
$$+\frac{1}{2} \int_\Omega \varphi_i \left[u_{1,j} \frac{\partial \varphi_j}{\partial x} + u_{2,j} \frac{\partial \varphi_j}{\partial y} \right] \left(\sum_{k=1}^{N} (u_l)_k \varphi_k \right), \; i = 1, \cdots, N$$

To solve the nonlinear system (3.47) we use the Newton-Raphson algorithm. Newton (or Newton-Raphson) method has quadratic convergence and it is one of the most common iterative method for solving nonlinear system of the form

$$\begin{aligned} h_1(x_1, \quad \cdots \quad, x_d) &= 0 \\ \cdots \quad \cdots \quad \cdots & \\ h_d(x_1, \quad \cdots \quad, x_d) &= 0 \end{aligned} \tag{3.49}$$

where h_i can be assumed as mapping a vector $\mathbf{x} = (x_1, \cdots, x_d)^t$ of $\mathbb{R}^d$ $(d = 2, 3)$ into $\mathbb{R}$.

Defining a function $\mathbf{H} = (h_1, \cdots, h_d)$ mapping $\mathbb{R}^d$ into $\mathbb{R}^d$ and using vector notation to represent the variables $x_1, \cdots, x_d$, the system (3.49) can be written as

$$\mathbf{H}(\mathbf{x}) = 0.$$

We define a function $\mathbf{G}$ by

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - \mathbf{J}(\mathbf{x})^{-1}\mathbf{H}(\mathbf{x})$$

where $\mathbf{J}$ is the Jacobian given by $\mathbf{J} = (J_{ij}) = \dfrac{\partial h_i}{\partial x_j}$, $i, j = 1, \cdots, d$ $(d = 2, 3)$.

Choosing initial value $\mathbf{x}^0$ and generating, for $k \geq 0$,

$$\begin{aligned} \mathbf{x}^{k+1} &= \mathbf{G}(\mathbf{x}^K) = \mathbf{x}^k - \mathbf{J}(\mathbf{x}^k)^{-1}\mathbf{H}(\mathbf{x}^k) \\ \Leftrightarrow \mathbf{J}\left(\mathbf{x}^{k+1} - \mathbf{x}^k\right) &= -\mathbf{H}(\mathbf{x}^k), \text{ for } k = 0, 1, \ldots \end{aligned} \tag{3.50}$$

which avoid explicit computation of $\mathbf{J}(\mathbf{x}^k)^{-1}$.

Taking $\mathbf{w}^k = \mathbf{x}^k - \mathbf{x}^{k+1}$, we solve the linear system defined by

$$\mathbf{J}\mathbf{w}^k = \mathbf{H}(\mathbf{x}^k)$$

in order to $\mathbf{w}^k$ and update $\mathbf{w}^{k+1} = \mathbf{x}^k - \mathbf{w}^k$.

For Navier-Stokes system, in fact, we want to solve the nonlinear vector field function $\mathbf{H}(\mathbf{u}, p) = 0$, where

$$\mathbf{H}(\mathbf{u}, p) = \begin{bmatrix} \nu\mathbf{A} & 0 & \mathbf{B}_x \\ 0 & \nu\mathbf{A} & \mathbf{B}_y \\ \mathbf{B}_x^t & \mathbf{B}_y^t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} c(\mathbf{u}_1) \\ c(\mathbf{u}_2) \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ 0 \end{bmatrix} \tag{3.51}$$

Considering the initial data $\mathbf{u}^0, p^0$ are known, we obtain

$$\begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{n} \\ p^{n} \end{bmatrix} - \mathbf{J}^{-1}(\mathbf{u}^n, p^n)\mathbf{H}(\mathbf{u}^n, p^n), \;\; n \geq 0 \tag{3.52}$$

Considering $\mathbf{J}^{-1}(\mathbf{u}^n, p^n)\mathbf{H}(\mathbf{u}^n, p^n) = \begin{bmatrix} \delta\mathbf{u}^n \\ \delta p^n \end{bmatrix}$[3], we have

$$\begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{n} \\ p^{n} \end{bmatrix} - \begin{bmatrix} \delta\mathbf{u}^{n} \\ \delta p^{n} \end{bmatrix} \tag{3.53}$$

So, we can define the algorithm

1. Given $(\mathbf{u}^0, p^0) \in \mathbf{V} \times Q$.

2. Repeat

$$\text{Solve } \mathbf{J} \begin{bmatrix} \delta\mathbf{u}^n \\ \delta p^n \end{bmatrix} = \mathbf{H}(\mathbf{u}^n, p^n)$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \delta\mathbf{u}^n$$

$$p^{n+1} = p^n - \delta p^n$$

until $\|((\delta\mathbf{u}^n, \delta p^n)\| < TOL$.

[3] $\mathbf{J}^{-1}(\mathbf{u}^n, p^n)\mathbf{H}(\mathbf{u}^n, p^n) = \begin{bmatrix} \delta\mathbf{u}^n \\ \delta p^n \end{bmatrix} \Leftrightarrow \mathbf{J} \begin{bmatrix} \delta\mathbf{u}^n \\ \delta p^n \end{bmatrix} = \mathbf{H}(\mathbf{u}^n, p^n)$

3.6 Numerical Results

In this section we are interested in the implementation of the iterative Newton-Raphson method to obtain the numerical solution of the steady Navier-Stokes equations. The Navier-Stokes problem was discretized using the $\mathbb{P}_2 - \mathbb{P}_1$ (Hood-Taylor) elements, to guarantee the stability. The numerical simulation was implemented on the general finite element solver FreeFem++ and we use the default solver sparse-solver [22] to solve the linear system (for more details about the solver we can see [41]). In fact, all the meshes and simulations were done using FreeFem++.

3.6.1 FreeFem++

FreeFem++ [22] is a free partial differential equations solver using finite element method with its own language. FreeFem++ documentation is accessible on www.freefem.org/ff++/ftp/FreeFem++doc.pdf. This software was developed in C++ at the Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris by Frédéric Hecht in collaboration with OlivierPironneau, Jacques Morice, Antoine Le Hyaric and Kohji Ohtsuka. The FreeFem++ language allows for a quick specification of any partial differential system of equations with the variational formulation of a linear steady state problem and the user need to write the own script. It is possible to solve coupled problems as we do to solve Oldroyd-B problem. FreeFem++ has an advanced automatic mesh generator, based on the Delaunay-Voronoi algorithm where the number of inner points is proportional to number of points on the boundaries, capable of posteriori mesh adaptation. It is also possible to read the mesh from external file and save the mesh to be used for other codes. It has a several triangular finite elements, including discontinuous elements. To solve linear or bilinear variational formulation, the user needs to parametrize the boundary domain and defines the number of nodes on boundary, even as boundary conditions and the variational form of PDE. He needs to define also the finite element type to use. For nonlinear forms the user needs to implement the method to apply for solving the problem. The FreeFem++ has some solver to linear systems.

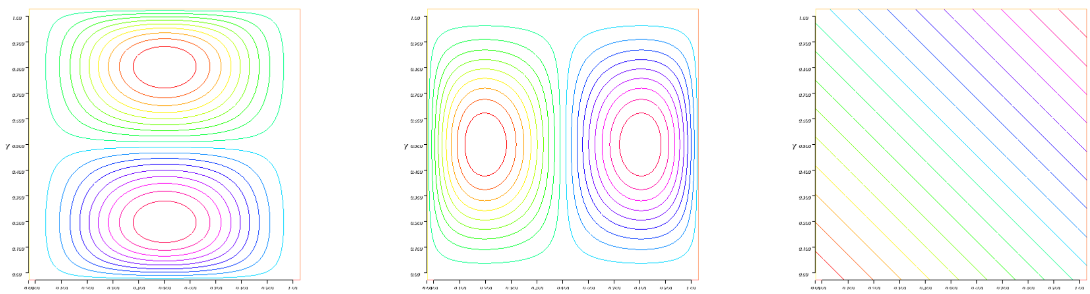
3.6.2 Validation of the Code

We developed our own script in FreeFem++ to implement the Newton method applied to the non-dimensional Navier-Stokes problem

$$\begin{cases} Re(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \Delta\mathbf{u} = \mathbf{f}, & \text{in } \Omega \\ \\ \mathbf{u} = \mathbf{u}_0, & \text{on } \partial\Omega. \end{cases} \tag{3.54}$$

To validate our code we fix the velocity and pressure

$$\begin{aligned} \mathbf{u}(x,y) &= \big((x^2-x)^2(y^2-y)(2y-1), -(x^2-x)(y^2-y)^2(2x-1)\big) \\ p(x,y) &= x+y \end{aligned} \tag{3.55}$$



(a) $u_1(x,y)=(x^2-x)^2(y^2-y)(2y-1)$ (b) $u_2(x,y)=-(x^2-x)(y^2-y)^2(2x-1)$ (c) $p(x,y)=x+y$

Figure 3.3: *Contour of the first component of velocity (on the left), second component of velocity (on the center) and pressure (on the right).*

and we evaluate the external forces $\mathbf{f} = (f_1, f_2)$ to verifies the Navier-Stokes equations with $Re = 1$.

$$\begin{aligned} f_1(x,y) = {} & -2(2x-1)^2(y^2-y)(2y-1) - 4(x^2-x)(y^2-y)(2y-1) - 6(x^2-x)^2(2y-1) \\ & +1 + 2(x^2-x)^3(y^2-y)^2(2y-1)^2(2x-1) - (x^2-x)(y^2-y)^2 \\ & (2x-1)\left[(x^2-x)^2(2y-1)^2 + 2(x^2-x)^2(y^2-y)\right] \end{aligned}$$

$$\begin{aligned} f_2(x,y) = {} & 6(2x-1)(y^2-y)^2 + 2(x^2-x)(2y-1)^2(2x-1) + 4(x^2-x)(y^2-y)(2x-1) \\ & +1 + (x^2-x)^2(y^2-y)(2y-1)\left[-(2x-1)^2(y^2-y)^2 - 2(x^2-x)(y^2-y)^2\right] \\ & +2(x^2-x)^2(y^2-y)^3(2x-1)^2(2y-1) \end{aligned}$$

We consider that the fluid is contained on a squared domain $\Omega = [0,1]^2$ and the Dirichlet boundary conditions prescribed agree with the exact solution according to (3.55).

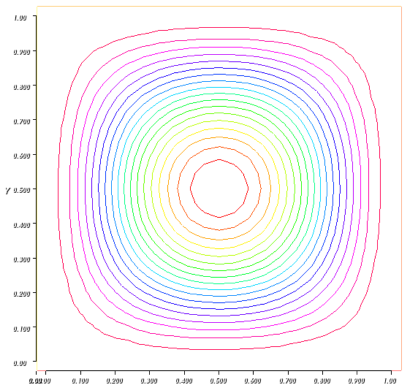
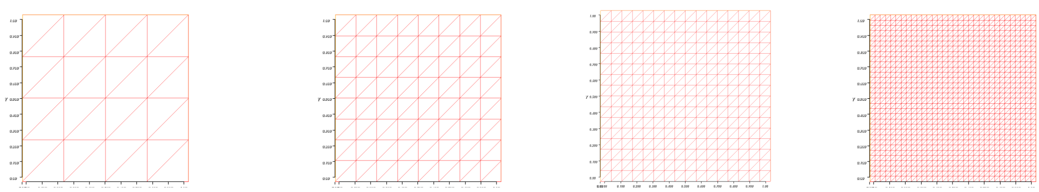


Figure 3.4: *Exact streamline.*

As we can see by the plot of the stream function, the fluid is rotating inside the domain with the same speed.

To guarantee the quadratic convergence of Newton's method applied to Navier-Stokes equations, we should choose an initial approximation nearby the exact solution. If we choose the initial approximation as the finite-element solution of Stokes equations, then the Newton's sequence converges quadratically to the unique solution to Navier-Stokes equations for sufficiently small mesh size h and a moderate Reynolds number Re [26]. The problem has been solved using four grids obtained by successive refinements dividing each triangle into four new triangles starting with a coarse mesh with 32-elements.



(a) 32 elements (b) 128 elements (c) 512 elements (d) 2048 elements

Figure 3.5: *Meshes over the square* $[0,1]^2$.

The Table 3.1 characterizes the mesh through the diameter h, number of elements, degree of freedoms.

Grid	h	No. of elements	$\mathbb{P}_2$ nodes	$\mathbb{P}_1$ nodes
Grid1	0.353553	32	81	25
Grid2	0.176777	128	289	81
Grid3	0.0883883	512	1089	289
Grid4	0.0441942	2048	4225	1089

Table 3.1: Characterizations of the grids

In each case, we evaluate the error of fluid velocity in $\mathbf{H}^1$-norm and the error of the pressure in L^2-norm which are respectively defined by

$$\begin{aligned} err_u &= \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} = \sum_{i=1}^{2} \|u_i - u_{h,i}\|_{H^1(\Omega)} \\ &= \sum_{i=1}^{2} \Big(\|u_i - u_{h,i}\|_{L^2(\Omega)} + \|\nabla(u_i - u_{h,i})\|_{L^2(\Omega)} \Big) \end{aligned}$$

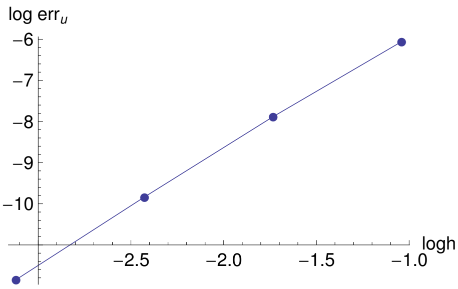
and $err_p = \|p - p_h\|_{L^2(\Omega)} = \left[\int_\Omega (p - p_h)^2 \right]^{1/2}$.

The results obtained for $\mathbf{u}$ and p over the different meshes are present in the following table (Table 3.2). The good convergence of results for all kinematic can be confirmed by the slope value. We used the least squares approximation to find the slope of the log-log plot of the error of the velocity and pressure.

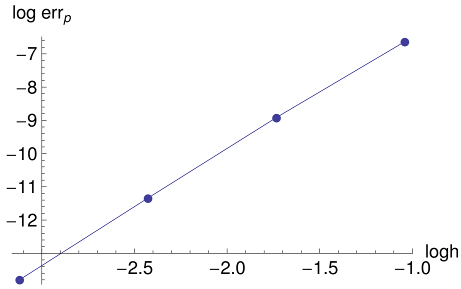
Error	Grid1	Grid2	Grid3	Grid4	Slope of the log-log plot
err_u	0.00234436	0.000377856	5.3719×10^{-5}	7.18033×10^{-6}	2.78906
err_p	0.00132182	0.000134722	1.19493×10^{-5}	1.02202×10^{-6}	3.45345

Table 3.2: Error of the velocity field and pressure

The following plots show us the error curves:



(a) Log-log plot of the error err_u

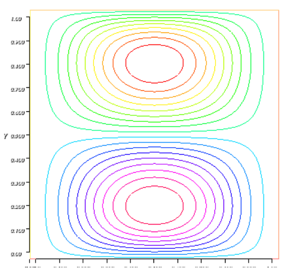


(b) Log-log plot of the error err_p

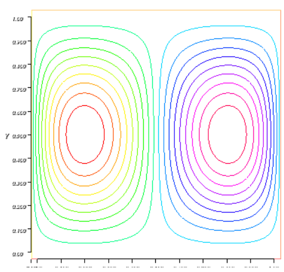
Figure 3.6: *Log-log plot of the error of the velocity and pressure.*

Like our expectation, the rate of convergence (the slope) is positive (quadratic for the velocity) for both the errors, and since the errors approaches zero as h tends to zero, so, our approximation converges to the exact solution with respect to the corresponding norms.

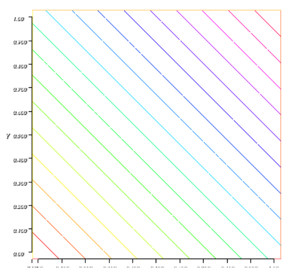
The approach solution is illustrated in the next figure:



(a) $u_{h,1}(x,y)\approx u_1(x,y)$



(b) $u_{h,2}(x,y)\approx u_2(x,y)$



(c) $p_h(x,y)\approx p(x,y)$

Figure 3.7: *Numerical solution obtained with* 512 *elements. Contour of the first component of velocity (on the left), second component of velocity (on the center) and pressure (on the right).*

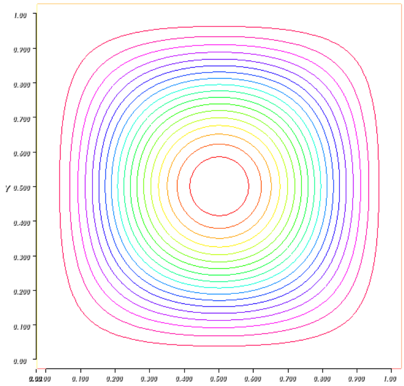


Figure 3.8: *Numerical streamline.*

To see the inertia effects (and consequentely the viscous effects, since there is a inverse relaton between the viscosity and the Reynolds number) in the behavior of the fluid, we consider the classical lid-driven cavity problem

$$\begin{cases} Re(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\Delta\mathbf{u}=0, \quad \text{in } \Omega \\ \\ u_1=1, u_2=0, \quad \text{on } \partial\Omega_3 \\ \\ u_1=0, u_2=0, \quad \text{on } \partial\Omega_1, \partial\Omega_2, \text{ and } \partial\Omega_4. \end{cases} \tag{3.56}$$

in the domain $\Omega=[0,1]^2$ with the boundary $\partial\Omega=\bigcup_{k=1}^{4}\partial\Omega_k$ as we illustrate in the next figure.

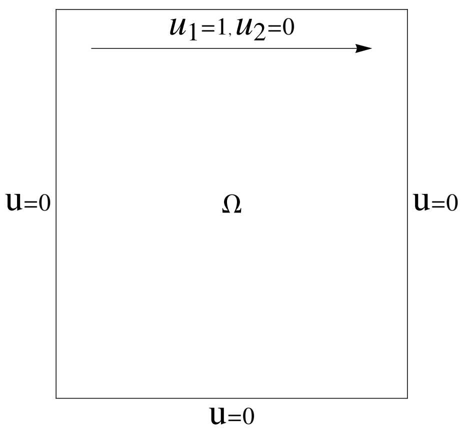


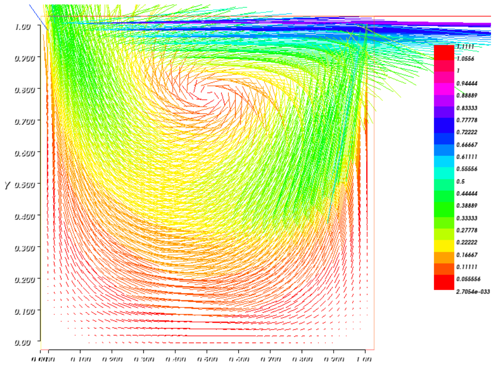
Figure 3.9: *Classical lid-driven cavity* Ω .

which has been investigated by many authors [44, 17] in the last thirty years. The problem considers the incompressible flow in a square domain (cavity) with an upper wall moving with a constant velocity $\mathbf{u} = \mathbf{1}$ from left to right. The other boundaries have null no-slip tangential and normal velocity boundary condition. The balance of viscous and pressure forces make the fluid turn into the square cavity. The properties of these forces depend upon the Reynold numbers.

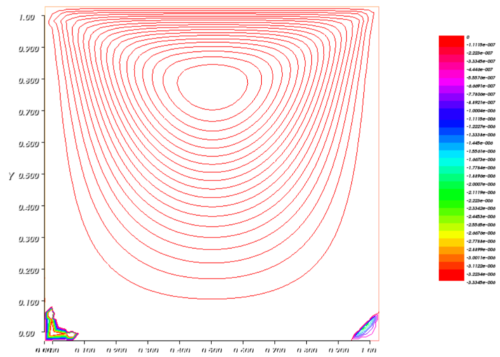
We solve the problem (3.56) for different Reynolds numbers discretizing the cavity such that each side is split into 16 parts. The resulting mesh has 512 elements, 1089 $\mathbb{P}_2$ nodes and 289 $\mathbb{P}_1$ nodes. For a better evaluation of the behavior of fluid flow we have computed the stream function ψ.

The next five figures[4] show the vector field and the stream function for results with different Reynolds. From the vector field we can see that the fluid rotates in the same direction of movement of the upper wall (from the left to the right). We observe from the stream function that the fluid rotates in the cavity with greater velocity close to the upper wall then the other parts. We can see the primary clockwise vortex, whose locations occurs towards the geometric centre of the square cavity. This vortex shifts progressively to the right as Re increases. This behaviour is a consequence of the viscosity effects. We can observe also the counter-clockwise rotating secondary eddies at the both corners close to the bottom.

[4] All the color scale are defined for 20 values equally spaced between its minimum (below) and the maximum (top) .

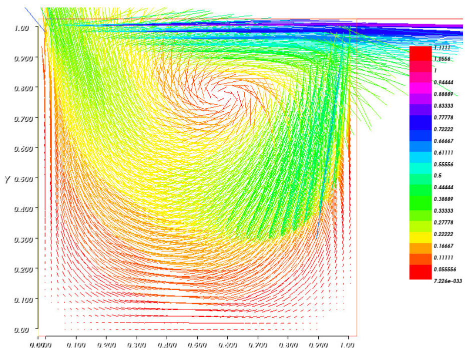


(a) Vector field at Re=1

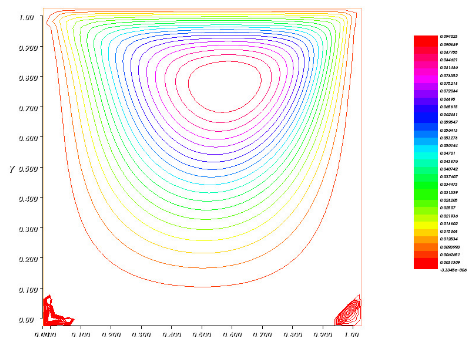


(b) Stream function at Re=1

Figure 3.10: *Vector field and stream function of the cavity flow problem with Reynolds number Re* $= 1$.

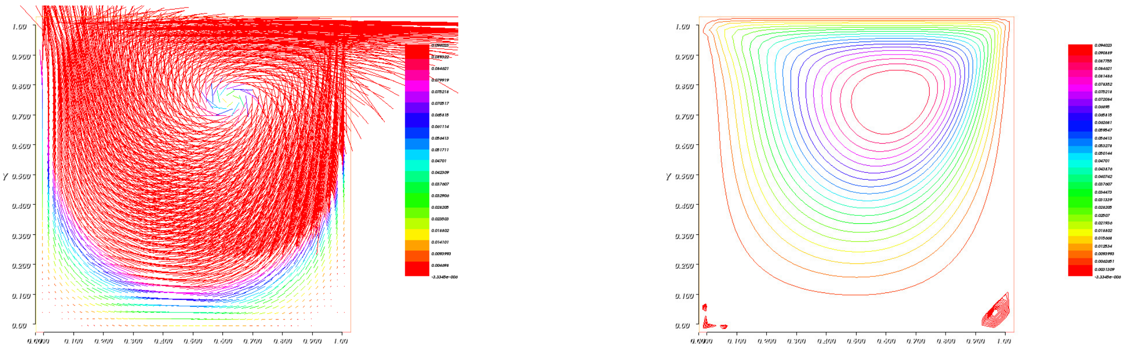


(a) Vector field at Re=50



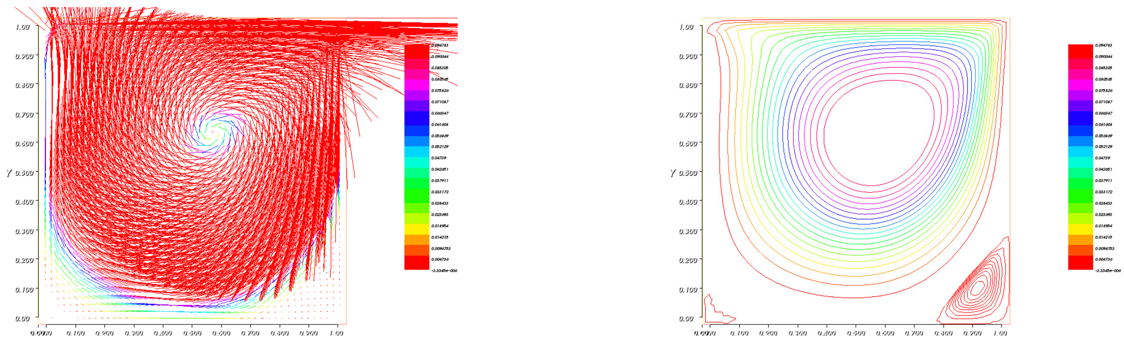
(b) Stream function at Re=50

Figure 3.11: *Vector field and stream function of the cavity flow problem with Reynolds number Re* $= 50$.



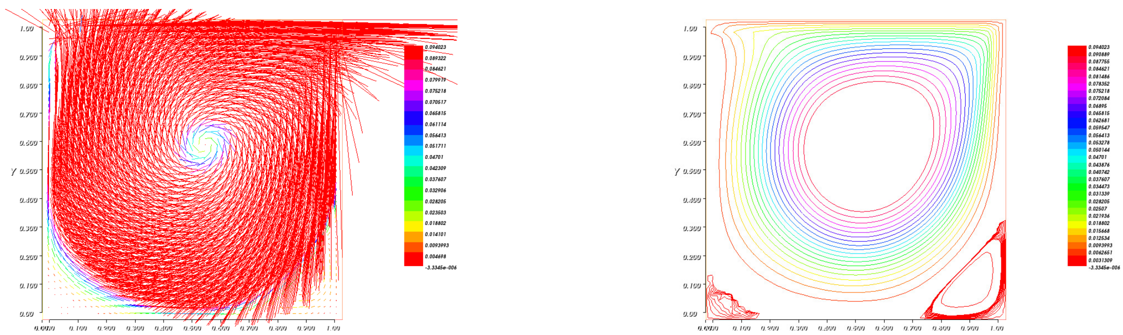
(a) Vector field at Re=100 (b) Stream function at Re=100

Figure 3.12: *Vector field and stream functionof the cavity flow problem with Reynolds number Re* $= 100$.



(a) Vector field at Re=300 (b) Stream function at Re=300

Figure 3.13: *Vector field and stream function of the cavity flow problem with Reynolds number Re* $= 300$.



(a) Vector field at Re=590 (b) Stream function at Re=590

Figure 3.14: *Vector field and stream function of the cavity flow problem with Reynolds number Re* $= 590$.

Chapter 4

Analysis of Steady Transport Problem

As an auxiliary problem for the steady Oldroyd-B model studied in the next chapter, we consider the steady tensorial transport equation.
In this chapter, we recall the essential results concerning the existence and uniqueness of solution as well as some considerations about the discretization of the problem in the application of discontinuous finite element method.

4.1 Mathematical Analysis for Steady Transport Problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded, open and connected Lipchitz domain. In this domain, we consider the steady tensorial transport equation, defined by

$$\boldsymbol{\sigma} + \lambda \mathbf{u} \cdot \nabla \boldsymbol{\sigma} = \mathbf{g} \tag{4.1}$$

where $\lambda \in L^\infty(\Omega)$, $\mathbf{u} \in \mathbf{L}^\infty(\Omega)$ and $\mathbf{g} \in \mathbf{L}^\infty(\Omega)$ are given.
In order to close this hyperbolic system and obtain a well-posed problem, the above equation need to be supplemented by boundary conditions on inflow sections of the boundary, according to the hyperbolic PDE theory.

Componentwise, the equation (4.1) can be written as

$$\begin{aligned} \sigma_{ij} + \lambda \mathbf{u} \cdot \nabla \sigma_{ij} &= g_{ij},\ i,j = 1,2 \Leftrightarrow \\ \frac{1}{\lambda}\sigma_{ij} + \mathbf{u} \cdot \nabla \sigma_{ij} &= \frac{1}{\lambda} g_{ij},\ \lambda \neq 0. \end{aligned} \tag{4.2}$$

Without loss of generality we can take $\lambda \neq 0$, because if $\lambda = 0$, we have $\boldsymbol{\sigma} = \mathbf{g}$ and $\mathbf{g}$ is given.

Assuming $\mu = \frac{1}{\lambda}$, the componentwise equation is scalar and can be identified as the hyperbolic equation known as advection-reaction equation

$$\mu w + \mathbf{u} \cdot \nabla w = h$$

4.1.1 Advection-reaction equation

For simplicity, we consider the steady advection-reaction equation with inflow homogeneous boundary condition[1]

$$\begin{cases} \mu w + \mathbf{u} \cdot \nabla w = h & \text{in } \Omega \\ \\ \mathbf{u} = 0 & \text{on } \partial\Omega^- \end{cases} \tag{4.3}$$

where $\mu \in L^\infty(\Omega)$ is the reaction coefficient, $\mathbf{u} \in \mathbf{L}^\infty(\Omega)$ is the advective velocity field, $h \in L^2(\Omega)$ is the source term, w is the unknown scalar function and $\partial\Omega^-$ denotes the inflow part of the boundary of Ω, namely

$$\partial\Omega^- = \{\mathbf{x} \in \partial\Omega : \mathbf{u}(\mathbf{x}).\mathbf{n}(\mathbf{x}) < 0\} \tag{4.4}$$

with $\mathbf{n} = (n_1, \cdots, n_d)^t$ be a unit outward normal to $\partial\Omega$.

In the similar way, we define the outflow part of $\partial\Omega$ as

$$\partial\Omega^+ = \{\mathbf{x} \in \partial\Omega : \mathbf{u}(\mathbf{x}).\mathbf{n}(\mathbf{x}) > 0\} \tag{4.5}$$

and the interior of the set $\{\mathbf{x} \in \partial\Omega : \mathbf{u}(\mathbf{x}).\mathbf{n}(\mathbf{x}) = 0\}$ as

$$\partial\Omega^0 = \partial\Omega \backslash \left(\overline{\partial\Omega^-} \cup \overline{\partial\Omega^+}\right).$$

[1] The extension to nonhomogeneous Dirichlet boundary condition on the sufficiently regular data being straight-forward

We assume that the inflow and outflow boundaries are well separated, i.e.,

$$\text{dist}\left(\partial\Omega^{-}, \partial\Omega^{+}\right) > 0.$$

We assume the following additional hypothesis on μ:

there exists $\mu_0 > 0$ such that

$$\mu(\mathbf{x}) - \frac{1}{2}\nabla\cdot\mathbf{u}(\mathbf{x}) \geq \mu_0 > 0 \ a.e. \text{ in } \Omega. \tag{4.6}$$

To obtain the weak formulation of (4.3) we introduce the graph space

$$W = \{w \in L^2(\Omega) : \mathbf{u}\cdot\nabla w \in L^2(\Omega)\} \subset L^2(\Omega) \tag{4.7}$$

Lemma 4.1.1

W is Hilbert space with respect to the graph norm

$$\|w\|_W = \|w\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla w\|_{L^2(\Omega)}.$$

□

Proof

Let v_n be an arbitrary Cauchy sequence in W. So, by (4.7), v_n and $\mathbf{u}\cdot\nabla v_n$ are the Cauchy sequence in $L^2(\Omega)$. Let the corresponding limits of v_n and $\mathbf{u}\cdot\nabla v_n$ are v and w in $L^2(\Omega)$. Let $\phi \in \mathcal{D}(\Omega)$. Integrating by parts we have

$$\int_\Omega w\phi \underset{n\to\infty}{\longleftarrow} \int_\Omega \mathbf{u}\cdot\nabla v_n\phi = -\int_\Omega v_n\nabla\cdot(\mathbf{u}\phi) \underset{n\to\infty}{\longrightarrow} -\int_\Omega v\nabla\cdot(\mathbf{u}\phi) = \int_\Omega \mathbf{u}\cdot\nabla v\phi$$

By the unicity of limit we conclude

$$\int_\Omega \mathbf{u}\cdot\nabla v\phi = \int_\Omega w\phi \Leftrightarrow \int_\Omega (\mathbf{u}\cdot\nabla v - w)\,\phi = 0 \ \forall\ \phi \in \mathcal{D}(\Omega).$$

By the density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$, we conclude that

$$\mathbf{u}\cdot\nabla v - w = 0 \text{ a.e. in } \Omega \Leftrightarrow \mathbf{u}\cdot\nabla v = w \text{ a.e. in } \Omega$$

then $\mathbf{u}\cdot\nabla v \in L^2(\Omega)$ and

$$\|v_n\|_W = \|v_n\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla v_n\|_{L^2(\Omega)} \longrightarrow \|v\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla v\|_{L^2(\Omega)} = \|v\|_W$$

So, W is a Hilbert space with the graph norm. ■

Remark

W is dense in $L^2(\Omega)$ and $H^1(\Omega)$ is a subspace of W.

Lesaint [27] guarantees the solution of (4.3) in the following results that he proves:

Theorem 4.1.1

Assume that $\mu \in L^\infty(\Omega)$ and $h \in L^2(\Omega)$. Then problem (4.3) has a unique strong solution $u \in W$. □

To specify mathematically the meaning of the boundary condition, we need to define the trace on $\partial\Omega$ of function in W. For that, we introduce the real-valued functions which are square integrable with respect to the measure $|\mathbf{u} \cdot \mathbf{n}|ds$, where ds is the Lebesgue measure on $\partial\Omega$, i.e.,

$$L^2\left(\partial\Omega; |\mathbf{u} \cdot \mathbf{n}|\right) = \{v \text{ is measurable on } \partial\Omega : \int_{\partial\Omega} |\mathbf{u} \cdot \mathbf{n}| v^2 ds < \infty\}.$$

The following lemma defines traces of functions belonging to W and the integration by parts formula [15].

Lemma 4.1.2 *(Traces and integration by parts)*

Suppose that $C^1(\overline{\Omega})$ is dense in W and $dist(\partial\Omega^-, \partial\Omega^+) > 0$, then the trace operator

$$\begin{aligned} \gamma : C^1(\overline{\Omega}) &\longrightarrow L^2\left(\partial\Omega; |\mathbf{u} \cdot \mathbf{n}|\right) \\ w &\longrightarrow \gamma(w) = w|_{\partial\Omega} \end{aligned}$$

extends uniquely to W, meaning that there is C_γ such that for all $v \in W$

$$\|\gamma(v)\|_{L^2(\partial\Omega; |\mathbf{u} \cdot \mathbf{n}|)} \le C_\gamma \|v\|_W$$

Moreover, the following integration by parts formula holds:

$$\int_\Omega [(\mathbf{u} \cdot \nabla w)v + (\mathbf{u} \cdot \nabla v)w + (\nabla \cdot \mathbf{u})wv] = \int_\Omega (\mathbf{u} \cdot \mathbf{n})wv$$

□

Let us introduce the following bilinear form

$$a(w,v) = \int_\Omega \mu wv + \int_\Omega (\mathbf{u}\cdot\nabla w)v + \int_{\partial\Omega} \frac{1}{2}\left(|\mathbf{u}\cdot\mathbf{n}| - \mathbf{u}\cdot\mathbf{n}\right) wv \quad \forall w,v\in W \tag{4.8}$$

Lemma 4.1.3

The bilinear form defined in (4.8) *is continuous and L^2-coercive in $W\times W$.* □

Proof

Let $w, v \in W$ be arbitrary functions. So,

$$\begin{aligned}
|a(w,v)| &= \left|\int_\Omega \mu wv + \int_\Omega (\mathbf{u}\cdot\nabla w)v + \frac{1}{2}\int_{\partial\Omega}\left(|\mathbf{u}\cdot\mathbf{n}| - \mathbf{u}\cdot\mathbf{n}\right) wv\right| \\
&\le \left|\int_\Omega (\mu w + \mathbf{u}\cdot\nabla w)\, v\right| + \frac{1}{2}\left|\int_{\partial\Omega}\left(|\mathbf{u}\cdot\mathbf{n}| - \mathbf{u}\cdot\mathbf{n}\right) wv\right| \\
&\le \left|\int_\Omega (\mu w + \mathbf{u}\cdot\nabla w)\, v\right| + \frac{1}{2}\left|\int_{\partial\Omega}|\mathbf{u}\cdot\mathbf{n}|\, wv\right| + \frac{1}{2}\left|\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n} wv\right| \\
&\le \int_\Omega |(\mu + \mathbf{u}\cdot\nabla)\, wv| + \int_{\partial\Omega}|\mathbf{u}\cdot\mathbf{n}|\,|wv| \\
&\underbrace{\le}_{\substack{trace\ theorem \\ Cauchy-Schwartz}} \|(\mu + \mathbf{u}\cdot\nabla)\, w\|_{L^2(\Omega)}\,\|v\|_{L^2(\Omega)} + c_\gamma\|w\|_W\|v\|_W \\
&\le \sqrt{2\ \max\{1, \|\mu\|^2_{L^\infty(\Omega)}\}}\|w\|_W\|v\|_{L^2(\Omega)} + c_\gamma\|w\|_W\|v\|_W
\end{aligned}$$

Indeed by parallelogram law, we have

$$\begin{aligned}
\|\mu w + \mathbf{u}\cdot\nabla w\|^2_{L^2(\Omega)} &= 2\,\|\mu w\|^2_{L^2(\Omega)} + 2\,\|\mathbf{u}\cdot\nabla w\|^2_{L^2(\Omega)} - \|\mu w - \mathbf{u}\cdot\nabla w\|^2_{L^2(\Omega)} \\
&\le 2\,\|\mu w\|^2_{L^2(\Omega)} + 2\,\|\mathbf{u}\cdot\nabla w\|^2_{L^2(\Omega)} \\
&\le 2\|\mu\|^2_{L^\infty(\Omega)}\|w\|^2_{L^2(\Omega)} + 2\,\|\mathbf{u}\cdot\nabla w\|^2_{L^2(\Omega)} \\
&\le 2\ \max\{1, \|\mu\|^2_{L^\infty(\Omega)}\}\left(\|w\|^2_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla w\|^2_{L^2(\Omega)}\right) \\
&= 2\ \max\{1, \|\mu\|^2_{L^\infty(\Omega)}\}\|w\|^2_W \\
\text{So, } \|\mu w + \mathbf{u}\cdot\nabla w\|_{L^2(\Omega)} &\le \sqrt{2\ \max\{1, \|\mu\|^2_{L^\infty(\Omega)}\}}\|w\|_W
\end{aligned}$$

Hence $a(.,.)$ is continuous. To prove L^2-coercivity, we take $w \in W$ an arbitrary function and by using integration by parts and (4.6), we have

$$\begin{aligned}
a(w,w) &= \int_\Omega \mu w^2 + \int_\Omega \mathbf{u}\cdot\nabla w^2 + \int_{\partial\Omega} \frac{1}{2}\left(|\mathbf{u}\cdot\mathbf{n}| - \mathbf{u}\cdot\mathbf{n}\right) w^2 \\
&= \int_\Omega \mu w^2 + \frac{1}{2}\int_\Omega \mathbf{u}\cdot\nabla(w^2) + \int_{\partial\Omega} \frac{1}{2}\left(|\mathbf{u}\cdot\mathbf{n}| - \mathbf{u}\cdot\mathbf{n}\right) w^2 \\
&= \int_\Omega \mu w^2 - \frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{u}) w^2 + \frac{1}{2}\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n}) w^2 + \int_{\partial\Omega} \frac{1}{2}\left(|\mathbf{u}\cdot\mathbf{n}| - \mathbf{u}\cdot\mathbf{n}\right) w^2 \\
&= \int_\Omega \left(\mu - \frac{1}{2}\nabla\cdot\mathbf{u}\right) w^2 + \frac{1}{2}\int_{\partial\Omega} |\mathbf{u}\cdot\mathbf{n}|\, w^2 \\
&\underbrace{\geq}_{using (4.6)} \mu_0 \|w\|^2 + \frac{1}{2}\int_{\partial\Omega} |\mathbf{u}\cdot\mathbf{n}|\, w^2
\end{aligned}$$

So, $a(.,.)$ is L^2-coercive. ■.

Consider the variational problem defined by

$$\begin{cases} \text{Find } w \in W \text{ such that} \\ a(w,v) = (h,v)\ \forall v \in W \end{cases} \tag{4.9}$$

Theorem 4.1.2

Problem (4.9) is well-posed. □

Proof

It is an immediate consequence of the following Lax-Milgram theorem (theorem 4.1.3).

■

Theorem 4.1.3 *(Lax-Milgram Theorem)*

Let V be a real Hilbert space endowed with the norm $\|.\|$, $a : V\times V \longrightarrow \mathbb{R}$ be a bilinear form, and let $f : V \longrightarrow \mathbb{R}$ be a continuous linear form i.e., $f \in V'$ where V' denotes the dual space of V. Moreover, assume that $a(.,.)$ is continuous and V-elliptic or coercive.

Then the abstract variational problem

$$a(u,v) = f(v) \quad \forall v \in V$$

has a unique solution $u \in V$, and

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'}$$

where α is the coerciveness constant of $a(.,.)$. □

The proof can be found in [7].

Proposition 4.1.1

Problem (4.9) *is the variational formulation of* (4.3).

Moreover, if $w \in W$ *is the solution of* (4.9), *then*

$$\begin{aligned} \mu w + \mathbf{u} \cdot \nabla w &= h \quad a.e \ in \ \Omega \\ \mu &= 0 \quad a.e \ in \ \Omega \end{aligned}$$

i.e., w is a weak solution of (4.3). □

Remark

We observe that the boundary condition is weakly enforced in (4.9).

4.2 Discontinuous Galerkin Method

The first discontinuous Galerkin method for hyperbolic partial differential equations have been introduced in 1973 by Reed and Hill to simulate nutron transport problem. The analysis of abstract form for this discrete problem was done one year later by Lesaint and Raviart [27]. More recently, the discontinuous Galerkin method for hyperbolic equations had a significant development based on numerical fluxes [12].
Discontinuous Galerkin method can be viewed as finite element method, but with relaxed continuity at interelement boundaries. The essential idea of the method is derived from the fact that the shape functions can be chosen so that the field variable and/or its derivatives are discontinuous across the element boundaries. The effect of the boundary conditions are gradually propagate through element-by-element connection. This way it is possible to introduce a centering in a scheme that contains the integral over the edges, using the right and left values of the edge side, along the direction of flow.

4.2.1 Discrete transport problem

The details of this method can be found in [15, 28, 27]. To simplify, like we did for the Navier-Stokes problem, we consider Ω a polyhedra, because this way we can covered

Ω exactly by a mesh of polyhedral elements as we explained.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of non-degenerate, regular triangulation of Ω (see page 38). An edge F of $\overline{\Omega}$ is a mesh face if either one of the following two conditions is satisfied:

(i) there are distinct mesh elements of K_1 and K_2 such that $F = \partial K_1 \cap \partial K_2$; in such case, F is called an interface

(ii) there is $K \in \mathcal{T}_h$ such that $F = \partial K \cap \partial\Omega$; in such case, F is called a boundary face.

Let $\mathcal{F}_h^i$ be the set of interfaces and $\mathcal{F}_h^b$ be the set of boundary faces. We set

$$\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$$

Let v be scalar function defined on Ω and assume that v is smooth enough to admit on all $F \in \mathcal{F}_h^i$ a possible two-valued trace (the interior and exterior traces of v on F). This means that, for all $K \in \mathcal{F}_h$, the restriction $v\mid_K$ of v to the open set K can be defined up to the boundary ∂K. Then, for all $F \in \mathcal{F}_h^i$ and $a.e.$ $x \in F$, the average and the jump of v is defined, respectively, as

$$\{v\}_F(x) = \frac{v_{|K_1}(x) + v_{|K_2}(x)}{2}$$

$$[v]_F(x) = v_{|K_1}(x) - v_{|K_2}(x)$$

where $K_i, i = 1, 2$ are distinct mesh elements such that $F = \partial K_1 \cap \partial K_2$. If $\mathbf{v}$ is a vector function, the average and jump operators act componentwise on the function $\mathbf{v}$. To simplify, the subscript F and the variable x are omitted.

We consider the broken polynomial space

$$\mathbb{P}_1(\boldsymbol{\tau}_h) = \left\{ v \in L^2(\Omega) : \forall K \in \mathcal{T}_h, v\mid_K \in \mathbb{P}_1(K) \right\}$$

where $\mathbb{P}_1(K)$ is spanned by the restriction to K of polynomials in $\mathbb{P}_1$ (set of polynomials defined in $\mathbb{R}^d$ with degree ≤ 1).

We define the broken Sobolev spaces as

$$H^m(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : \forall K \in \mathcal{T}_h, v\mid_K \in H^m(K) \right\},\ m \geq 0 \text{ is an integer}$$

and the broken gradient ∇_h by

$$\begin{aligned} \nabla_h : H^1(\mathcal{T}_h) &\longrightarrow \mathbf{L}^2(\Omega) \\ v &\longrightarrow (\nabla_h v)\mid_K = \nabla(v\mid_K) \end{aligned} \tag{4.10}$$

Lemma 4.2.1

Broken gradient on usual Sobolev spaces.

Let $m \geq 0$. There holds $H^m(\Omega) \subset H^m(\mathcal{T}_h)$. Moreover, for all $v \in H^1(\Omega)$, $\nabla_h v = \nabla v$ in $\mathbf{L}^2(\Omega)$. □

Lemma 4.2.2

A function $v \in H^1(\mathcal{T}_h)$ belongs to $H^1(\Omega)$ if and only if

$$[v] = 0, \quad \forall F \in \mathcal{F}_h^i.$$

□

In the framework of the transport problem, we denote by S_1 the subspace of $L^2(\Omega)$ whose functions are piecewise linear polynomial functions over $\mathcal{T}_h$ with degree less or equal to 1.

If the exact solution w is regular, we hope that the approach solution will be regular also [43]. In this sense, we added to the classical formulation a priori small term

$$\int_{\partial K} \left(\alpha \left| \mathbf{u} \cdot \mathbf{n}_K \right| - \frac{1}{2} \mathbf{u} \cdot \mathbf{n}_K \right) [w_h]\, v_h$$

expressing the discontinuities of the solution approach to interfaces of elements.

So, we want to find $w_h \in S_1 : w_h = g_h$ on $\partial\Omega^-$ such that

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K (\mu w_h + \mathbf{u} \cdot \nabla w_h)\, v_h + \int_{\partial K} \left(\alpha \left| \mathbf{u} \cdot \mathbf{n}_K \right| - \frac{1}{2} \mathbf{u} \cdot \mathbf{n}_K \right) [w_h]\, v_h \\ - \int_{\partial\Omega^-} \left| \mathbf{u} \cdot \mathbf{n}_K \right| w_h \chi_{\partial\Omega^-} v_h = \int_K h v_h, \ \forall v_h \in S_1 : v_h = 0 \text{ on } \partial\Omega^- \end{aligned} \tag{4.11}$$

where g_h is an approach of g on $\partial\Omega^-$, $\mathbf{n}_K$ is the unit outward normal to K and $\chi_{\partial\Omega^-}$ denotes the characteristic function of $\partial\Omega^-$.

The parameter α can vary from face to face but the value $\alpha = \frac{1}{2}$ is used usually according to the literature. With this value, one obtains the DG method analyzed by Lesaint and Raviart [27]. In this case the term $\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)[w_h]$ is non zero only on that part of the boundary ∂K where $\mathbf{u}\cdot\mathbf{n}_K < 0$.

4.3 Numerical Results

In this section, we are interested in the implementation of the iterative method based on the application of a fixed point method in FreeFem+ to solve the transport equation.

4.3.1 Validation of the code

We develop our own script in FreeFem++ to obtain the numerical solution of the non-dimensional steady tensorial transport equation using $\mathbb{P}_1$ discontinuous finite element ($\mathbb{P}_1 dc$). For this type of elements, due to interpolation problem, FreeFem++ doesn't consider the degree of freedom as the vertices but three vertices move inside on the element with the linear map $T(\mathbf{X}) = \mathbf{G} + 0.99(\mathbf{X} - \mathbf{G})$ where $\mathbf{G}$ is the barycenter. In this way, the number of degree of freedom is 3 times of the number of elements. Consider the following auxiliary problem of Oldroyd-B problem $(2.38)_3$ defined in $\Omega = [0,1]^2$:

find $\boldsymbol{\sigma} \in \mathbf{L}^2(\Omega))$ such that

$$\boldsymbol{\sigma} + We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] = 2\lambda\mathbf{D}(\mathbf{u}) + We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right] \quad \text{in} \;\; \Omega. \tag{4.12}$$

where the vector field $\mathbf{u}$ is given by (see [5])

$$\mathbf{u}(x,y) = \left((x^2-x)^2(y^2-y)(2y-1), -(x^2-x)(y^2-y)^2(2x-1)\right) \tag{4.13}$$

Using the discrete formulation, the dimensionless transport problem can be written in terms of its components as a system of three scalar transport equations (advection-reaction equations) ($\boldsymbol{\sigma}$ is symmetric tensor ($\sigma_{12} = \sigma_{21}$)), i.e., find $(\sigma_{11}, \sigma_{12}, \sigma_{22})$ such that

$$\begin{cases} \displaystyle\int_\Omega\left[\sigma_{h,11} + We\left(u_1\frac{\partial \sigma_{h,11}}{\partial x_1} + u_2\frac{\partial \sigma_{h,11}}{\partial x_2}\right)\right]\tau_{h,11} + We\int_{\partial K}\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)[\sigma_{h,11}]\tau_{h,11} \\ \displaystyle\quad = 2\lambda\int_\Omega \frac{\partial u_1}{\partial x_1}\tau_{h,11} + 2We\int_\Omega\left[\frac{\partial u_1}{\partial x_1}\sigma_{h,11} + \frac{\partial u_1}{\partial x_2}\sigma_{h,12}\right]\tau_{h,11} \\ \displaystyle\int_\Omega\left[\sigma_{h,12} + We\left(u_1\frac{\partial \sigma_{h,12}}{\partial x_1} + u_2\frac{\partial \sigma_{h,12}}{\partial x_2}\right)\right]\tau_{h,12} + We\int_{\partial K}\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)[\sigma_{h,12}]\tau_{h,12} \\ \displaystyle\quad = \lambda\int_\Omega\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\tau_{h,12} + We\int_\Omega\left[\frac{\partial u_2}{\partial x_1}\sigma_{h,11} + \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\sigma_{h,12} + \frac{\partial u_1}{\partial x_2}\sigma_{h,22}\right]\tau_{h,12} \\ \displaystyle\int_\Omega\left[\sigma_{h,22} + We\left(u_1\frac{\partial \sigma_{h,22}}{\partial x_1} + u_2\frac{\partial \sigma_{h,22}}{\partial x_2}\right)\right]\tau_{h,22} + We\int_{\partial K}\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)[\sigma_{h,22}]\tau_{h,22} \\ \displaystyle\quad = 2\lambda\int_\Omega \frac{\partial u_2}{\partial x_2}\tau_{h,22} + 2We\int_\Omega\left[\frac{\partial u_2}{\partial x_1}\sigma_{h,12} + \frac{\partial u_2}{\partial x_2}\sigma_{h,22}\right]\tau_{h,12} \end{cases} \tag{4.14}$$

For the computational implementation, we consider a mesh 100×100 with 20000 elements and 60000 $\mathbb{P}_1 dc$ nodes and we take $\lambda = 0.1, 0.5$ and 0.9. For each λ we do the study for different values of We admissible for convergence of iterative method. This iterative method can be described as follows

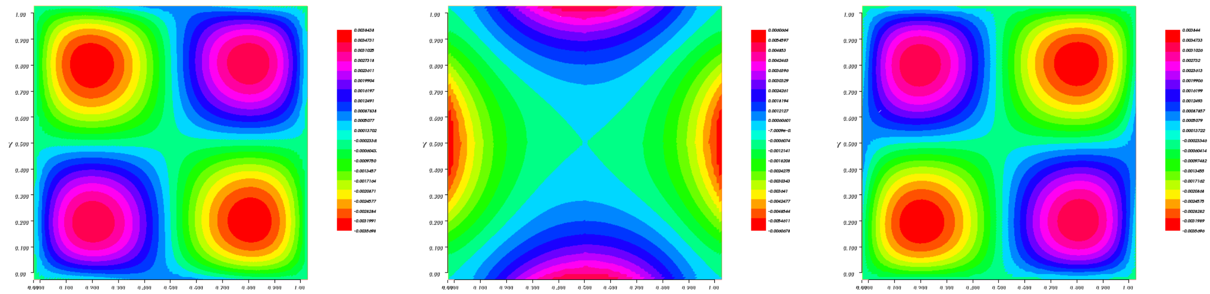
given $(\sigma^0_{11}, \sigma^0_{12}, \sigma^0_{22})$ such that

$$\begin{cases} \displaystyle\int_\Omega\left[\sigma^{n+1}_{h,11} + We\left(u_1\frac{\partial \sigma^{n+1}_{h,11}}{\partial x_1} + u_2\frac{\partial \sigma^{n+1}_{h,11}}{\partial x_2}\right)\right]\tau_{h,11} + We\int_{\partial K}\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)\left[\sigma^{n+1}_{h,11}\right]\tau_{h,11} \\ \displaystyle\quad = 2\lambda\int_\Omega \frac{\partial u_1}{\partial x_1}\tau_{h,11} + 2We\int_\Omega\left[\frac{\partial u_1}{\partial x_1}\sigma^n_{h,11} + \frac{\partial u_1}{\partial x_2}\sigma^n_{h,12}\right]\tau_{h,11} \\ \displaystyle\int_\Omega\left[\sigma^{n+1}_{h,12} + We\left(u_1\frac{\partial \sigma^{n+1}_{h,12}}{\partial x_1} + u_2\frac{\partial \sigma^{n+1}_{h,12}}{\partial x_2}\right)\right]\tau_{h,12} + We\int_{\partial K}\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)\left[\sigma^{n+1}_{h,12}\right]\tau_{h,12} \\ \displaystyle\quad = \lambda\int_\Omega\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\tau_{h,12} + We\int_\Omega\left[\frac{\partial u_2}{\partial x_1}\sigma^n_{h,11} + \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\sigma^n_{h,12} + \frac{\partial u_1}{\partial x_2}\sigma^n_{h,22}\right]\tau_{h,12} \\ \displaystyle\int_\Omega\left[\sigma^{n+1}_{h,22} + We\left(u_1\frac{\partial \sigma^{n+1}_{h,22}}{\partial x_1} + u_2\frac{\partial \sigma^{n+1}_{h,22}}{\partial x_2}\right)\right]\tau_{h,22} + We\int_{\partial K}\left(\alpha\,|\mathbf{u}\cdot\mathbf{n}_K| - \frac{1}{2}\mathbf{u}\cdot\mathbf{n}_K\right)\left[\sigma^{n+1}_{h,22}\right]\tau_{h,22} \\ \displaystyle\quad = 2\lambda\int_\Omega \frac{\partial u_2}{\partial x_2}\tau_{h,22} + 2We\int_\Omega\left[\frac{\partial u_2}{\partial x_1}\sigma^n_{h,12} + \frac{\partial u_2}{\partial x_2}\sigma^n_{h,22}\right]\tau_{h,22} \end{cases} \tag{4.15}$$

The linear system was solved using the default solver sparsesolver.

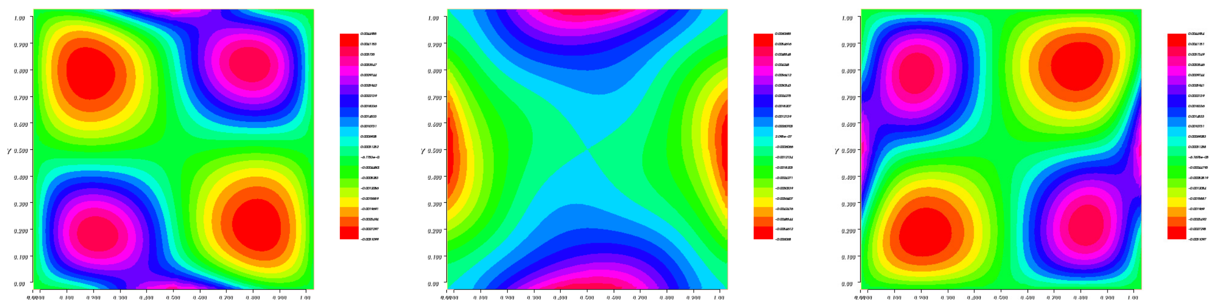
We analyze the effects of Weissenber number. It is known that the high We lead a numerical instabilities which can be seen in the behavior of numerical solution or lead the divergence of algorithm. We compared the different solutions obtained for We between 1 and 10 with fixed λ. For each λ, we observed the occurrence of numerical instabilities associated with the increased We as we can see with the next figures[2]. When λ increase, we observed that the qualitative behavior is the same but quantitatively the amplitude of each component increases. The following figures show the behavior of three components of tensor for different We and different λ.

[2]All the color scales are defined for 20 values equally spaced between its minimum (below) and the maximum (top) .



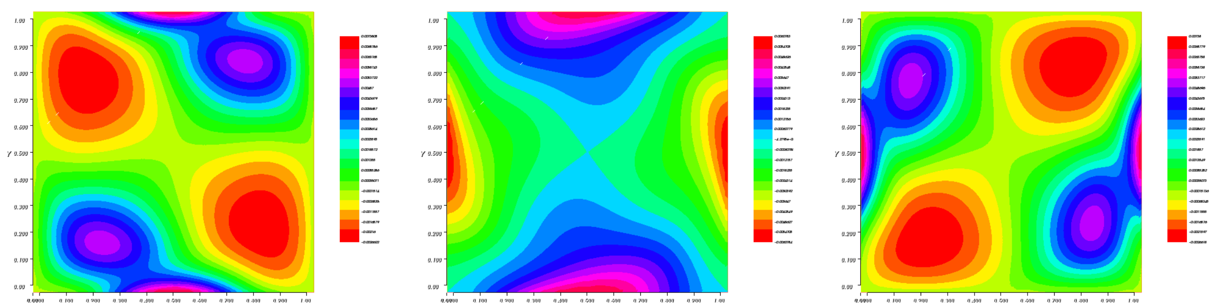
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.1: *Contours of the stress tensor components at* $We = 1$ *and* $\lambda = 0.1$.



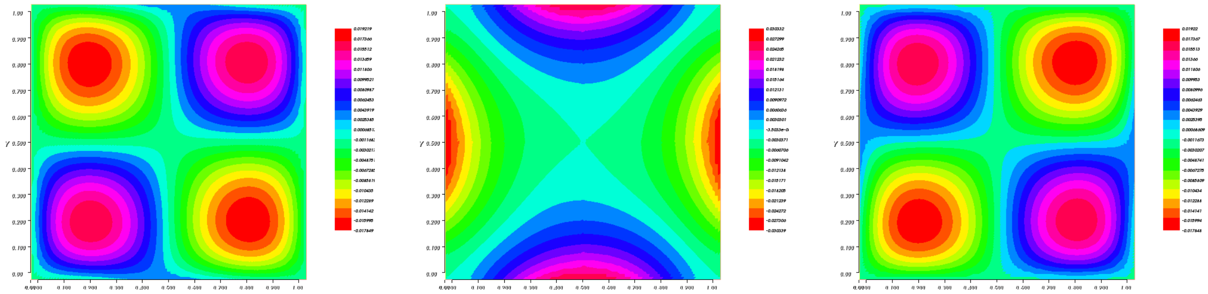
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.2: *Contours of the stress tensor components at* $We = 5$ *and* $\lambda = 0.1$.



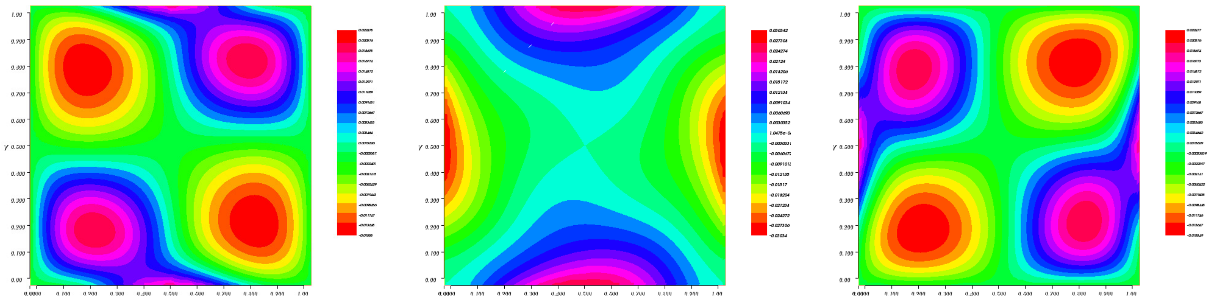
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.3: *Contours of the stress tensor components at* $We = 10$ *and* $\lambda = 0.1$.



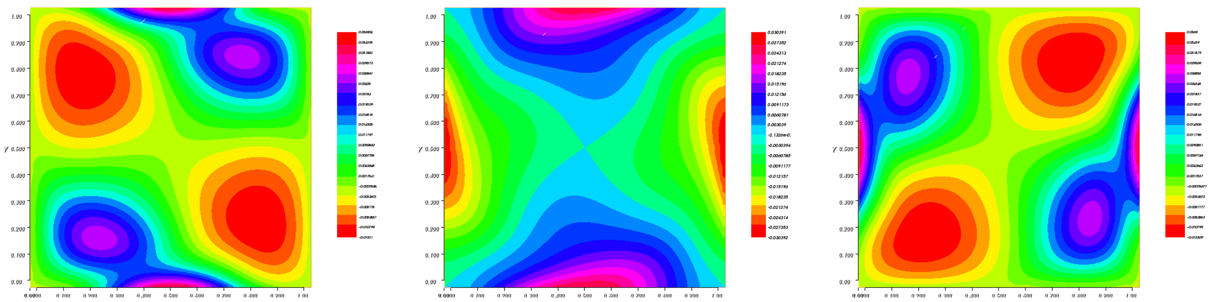
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.4: *Contours of the stress tensor components at* $We = 1$ *and* $\lambda = 0.5$.



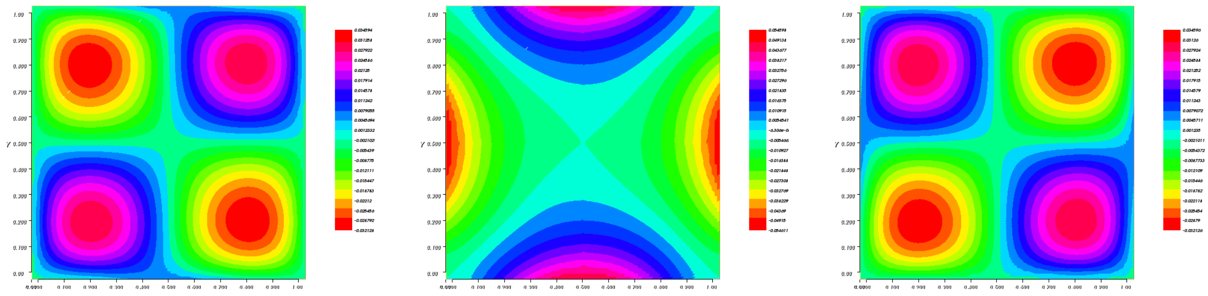
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.5: *Contours of the stress tensor components at* $We = 5$ *and* $\lambda = 0.5$.



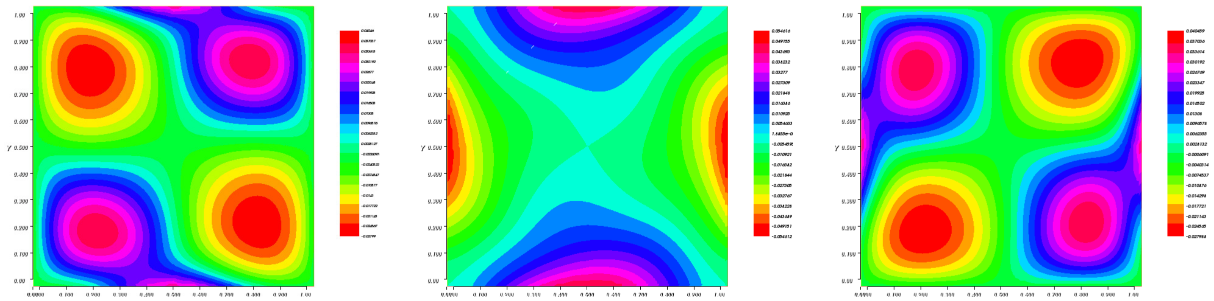
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.6: *Contours of the stress tensor components at* $We = 10$ *and* $\lambda = 0.5$.



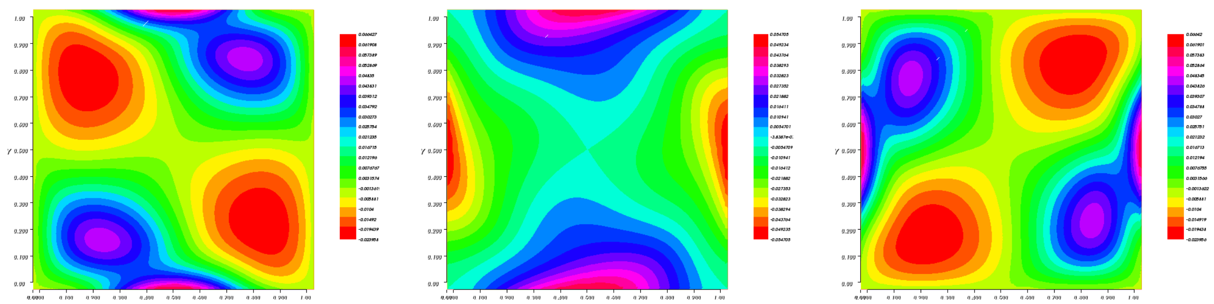
(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.7: *Contours of the stress tensor components at $We = 1$ and $\lambda = 0.9$.*



(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.8: *Contours of the stress tensor components at $We = 5$ and $\lambda = 0.9$.*



(a) σ_{11} (b) σ_{12} (c) σ_{22}

Figure 4.9: *Contours of the stress tensor components at $We = 10$ and $\lambda = 0.9$.*

Chapter 5

Oldroyd-B Fluids Flows

In this chapter, we want to study the steady problem which model the behavior of Oldroyd-B fluids type, in a bi-dimensional domain $\Omega \subset \mathbb{R}^2$. It is done the approach of Oldryod-B model and will be presented numerical results. We apply the fixed point method type proposed by Najib and Sandri [30].

As we refer in chapter 1, given $0 < \lambda < 1$, we want to approach the solution $(\mathbf{u}, p, \boldsymbol{\sigma})$ of the problem

find $(\mathbf{u}, \boldsymbol{\sigma}, p)$, defined in Ω such that

$$\begin{cases} Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \nabla p = (1-\lambda)\Delta\mathbf{u} + \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] + \boldsymbol{\sigma} = 2\lambda\mathbf{D}(\mathbf{u}) + We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right] \quad \text{in} \quad \Omega. \end{cases}$$

where Re and We are, respectively, the Reynolds number and Weissenberg number.

5.1 The Oldroyd-B Constitutive Equation

The Oldroyd-B constitutive equation in case of steady flow read in the form of (2.30) as

$$\begin{cases} \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu_n\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \lambda_1\,(\mathbf{u}\cdot\nabla)\,\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma}, \nabla\mathbf{u}) \quad \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

where $\mathbf{h}(\boldsymbol{\sigma}, \nabla \mathbf{u}) = 2\mu_e \mathbf{D}(\mathbf{u}) + \lambda_1 \left[(\nabla \mathbf{u})\sigma + \sigma(\nabla \mathbf{u})^t\right]$.

The above equation is composed of a Navier-Stokes like system for $(\mathbf{u}, p)$ and a transport equation for extra stress tensor $\boldsymbol{\sigma}$. The system of equations (5.1) have to be used with some boundary conditions. For a connected flow domain $\Omega \subset \mathbb{R}^2$, the required boundary conditions are the following:

(i) Dirichlet boundary conditions for the velocity on the boundary $\partial\Omega$

$$\mathbf{u} = \mathbf{g} \;\text{ on } \partial\Omega \text{ with compatibilty condition } \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0.$$

where $\mathbf{n}$ is the unit outward normal vector to Ω at the boundary $\partial\Omega$. For homogeneous case, $\mathbf{g} = 0$.

(ii) For the stress, a condition on the upstream boundary section

$$\begin{aligned} \partial\Omega^- &= \{\mathbf{x} \in \partial\Omega : \mathbf{u}(\mathbf{x}).\mathbf{n}(\mathbf{x}) < 0\} \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}_{\partial\Omega} \text{ on } \partial\Omega^-. \end{aligned}$$

With the homogeneous Dirichlet boundary conditions, the Oldroyd-B fluid model problem is well-posed. In case of non-homogeneous boundary conditions, the equations of motion for Oldroyd-B fluids have an infinite number of solutions [25, 21]. For this reason the inflow boundary condition (ii) and outflow boundary condition should be imposed in order to insure the well-posedness [36].

With the homogeneous Dirichlet boundary conditions defined over Ω, the problem of determining the extra stress tensor $\boldsymbol{\sigma}$, the velocity $\mathbf{u}$ and the pressure p satisfying the Oldroyd-B constitutive equations can be reformulated as follows:

Find the quantities $\boldsymbol{\sigma}, \mathbf{u}$ and p, defined in Ω such that

$$
\begin{cases}
\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu_n\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f}, \quad \text{in} \ \ \Omega \\
\nabla\cdot\mathbf{u} = 0, \quad \text{in} \ \ \Omega \\
\lambda_1\,(\mathbf{u}\cdot\nabla)\,\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma},\nabla\mathbf{u}), \quad \text{in} \ \ \Omega \\
\mathbf{u} = 0, \quad \text{on} \ \partial\Omega
\end{cases}
\tag{5.2}
$$

subjected to the boundary condition (ii).

For non-dimensional case, the problem (5.2) can be read as the form of (2.38) as follows:

Find the non-dimensional quantities, still denoted by $\boldsymbol{\sigma}$, $\mathbf{u}$ and p, defined in Ω such that

$$
\begin{cases}
Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \nabla p = (1-\lambda)\Delta\mathbf{u} + \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}, \quad \text{in} \ \ \Omega \\
\nabla\cdot\mathbf{u} = 0, \quad \text{in} \ \ \Omega \\
We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] + \boldsymbol{\sigma} = 2\lambda\mathbf{D}(\mathbf{u}) + We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right], \quad \text{in} \ \ \Omega.
\end{cases}
\tag{5.3}
$$

subjected to the boundary conditions in (i) and (ii).

5.2 Variational Formulation

We consider an incompressible viscoelastic fluid confined into a domain Ω with fixed boundary. Mathematically, we write the steady Oldroyd-B equations with the Dirichlet boundary conditions (to simplify) i.e., $\mathbf{u} = \mathbf{u}_0$ such that $\mathbf{u}_0\cdot\mathbf{n} = 0$ on $\partial\Omega$. So, given an external force field $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$ and $0<\lambda<1$ the viscoelastic fraction of the viscosity, the steady Oldroyd-B problem is defined by

$$
\begin{cases}
Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}] - (1-\lambda)\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} \ \ \text{in} \ \ \Omega, \\
\nabla\cdot\mathbf{u} = 0 \ \ \text{in} \ \ \Omega, \\
\boldsymbol{\sigma} + We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] = 2\lambda\mathbf{D}(\mathbf{u}) + We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right] \ \ \text{in} \ \ \Omega \\
\mathbf{u} = \mathbf{u}_0, \ \ \mathbf{u}_0\cdot\mathbf{n} = 0 \ \ \text{on} \ \ \partial\Omega.
\end{cases}
\tag{5.4}
$$

Consider the space $\mathbf{H}_0^1(\Omega)$. Taking $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ an arbitrary test function, and the scalar product between the momentum equation and $\mathbf{v}$, and integrating over Ω, we obtain

$$\int_\Omega Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}] - \int_\Omega (1-\lambda)\Delta\mathbf{u}\cdot\mathbf{v} + \int_\Omega \nabla p\cdot\mathbf{v} = \int_\Omega \nabla\cdot\boldsymbol{\sigma}\cdot\mathbf{v} + \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad (5.5)$$

Applying the Green's formula (theorem 3.1.3) to (5.5) and taking into account that $\mathbf{v}$ vanishes on the boundary, we have

$$\int_\Omega Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}] + \int_\Omega (1-\lambda)\nabla\mathbf{u} : \nabla\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} = -\int_\Omega \boldsymbol{\sigma} : \nabla\mathbf{v} + \int_\Omega \mathbf{f}\cdot\mathbf{v}$$

As we take a Dirichlet problem, the pressure is determined only up to constant, since it appears in the equations only through its gradient. So, we consider the space $L_0^2(\Omega)$ and we take $q \in L_0^2(\Omega)$. Multiplying the continuity equation by q and integrating over Ω, we have

$$\int_\Omega q\nabla\cdot\mathbf{u} = 0.$$

Due to conservation of momentum, the tensor need to be symmetric. So, taking $\boldsymbol{\tau} \in \mathbf{L}_s^2(\Omega)$ arbitrary and the scalar product between the transport equation and $\boldsymbol{\tau}$, we obtain by the Green formula

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} + We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] : \boldsymbol{\tau} - We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^T\right] : \boldsymbol{\tau} - 2\lambda\mathbf{D}(\mathbf{u}) : \boldsymbol{\tau} = 0$$

The variational form to Oldroyd-B problem reads

Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, find $(\mathbf{u}, p, \boldsymbol{\sigma}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{L}_s^2(\Omega)$ such that

$$\begin{cases} \int_\Omega Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}] + \int_\Omega (1-\lambda)\nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega \boldsymbol{\sigma} : \nabla\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} \\ \int_\Omega q\nabla\cdot\mathbf{u} = 0 \\ \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] : \boldsymbol{\tau} - \int_\Omega We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right] : \boldsymbol{\tau} = \int_\Omega 2\lambda\mathbf{D}(\mathbf{u}) : \boldsymbol{\tau} \end{cases} \quad (5.6)$$

for all $(\mathbf{v}, p, \boldsymbol{\tau}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{L}_s^2(\Omega)$

5.3 Existence and Uniqueness Results of Solutions

Several authors [2, 50, 20, 40, 39] proved some results concerning the existence and uniqueness of solutions of problem (5.2), under certain smallness and regularity conditions on the data, and using a fixed point argument related to the decoupling of the original problem.
Renardy [40, 39] has obtained existence of stationary solutions for any value of λ, the other parameter being small, using a fixed-point method.

Theorem 5.3.1
Suppose that Ω *is class* C^q, $q = max\{k+2, 2\}$ *with* $k \geq 1$ *integer. Let* $\frac{n}{2} < p < \infty$, $\mathbf{f} \in \mathbf{W}^{k,p}(\Omega)$, $\mathbf{u}_0 \in \mathbf{W}^{k+2-\frac{1}{p},p}(\partial\Omega)$ *such that* $\mathbf{u}_0 \cdot \mathbf{n} = 0$ *and* γ *a constant that verify*

$$\|\mathbf{f}\|_{\mathbf{W}^{k,p}(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{W}^{k+2-\frac{1}{p},p}(\partial\Omega)} < \gamma$$

then there are $\gamma_0 > 0, \lambda_0 \in]0,1[$ *such that for all* $0 < \gamma < \gamma_0$ *and* $0 < \lambda < \lambda_0$, *the Oldroyd-B problem* (5.4) *has a unique solution*

$$(\mathbf{u}, p, \boldsymbol{\sigma}) \in \mathbf{W}^{k+2,p}(\Omega) \times \left(W^{k+1,p}(\Omega) \cap L^p_0(\Omega)\right) \times \mathbf{W}^{k+1,p}_s(\Omega).$$

Moreover

$$\|\mathbf{u}\|_{\mathbf{W}^{k+2,p}(\Omega)} + \|\boldsymbol{\sigma}\|_{\mathbf{W}^{k+1,p}(\Omega)} + \|p\|_{W^{k+1,p}(\Omega)} \leq c\left(\|\mathbf{f}\|_{\mathbf{W}^{k,p}(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{W}^{k+2-\frac{1}{p},p}(\Omega)}\right)$$

where c *is a constant depending of* $\mathbf{n}, p$ *and* Ω. □

The proof can be found in [50].

5.4 Discrete Oldroyd-B Problem

In this section, we study the approximation of the problem (5.2) using finite element method and recall the notations already used in the previous chapters. The system (5.2) is a composed (coupled) problem for the three unknowns $(\mathbf{u}, p, \boldsymbol{\sigma})$. We will use iterative scheme to solve this system. If $\boldsymbol{\sigma}$ is fixed, the first two equations of (5.2) defines a Navier-Stokes systems in the variables $\mathbf{u}$ and p. As in chapter 2, we use the Hood-Taylor finite element method for the approximation of the velocity and the pressure field $(\mathbf{u}, p)$, where the corresponding spaces satisfy the discrete inf-sup condition. If $\mathbf{u}$ (and p) are fixed, then the third equation of (5.2) is a transport equation in the variable $\boldsymbol{\sigma}$. The approximation of $\boldsymbol{\sigma}$ will be done by using discontinuous Galerkin finite element method, as in chapter 3.

To obtain the approximate problem, we apply the finite element method. To approach the hyperbolic transport problem we consider the Discontinuous Galerkin Method and to approach the elliptic Navier-Stokes problem we consider the Hood-Taylor finite element method.

Consider the case where $0 < \lambda < 1$ (if $\lambda = 0$ we have a Newtonian model and if $\lambda = 1$ we have the Maxwell model).

Let $\mathcal{T}_h,\ h > 0$, where h is discretization parameter, be a non-degenerated regular triangulation of Ω such that $\overline{\Omega} = \bigcup\limits_{K\in\mathcal{T}_h} K$.

Consider the spaces $\mathbf{V}_h^0$ and M_h defined by (3.39) and (3.40) respectively, and the space

$$\mathbf{T}_h = \left\{\boldsymbol{\sigma}_h \in \mathbf{T} \cap \mathbf{C}(\Omega) \mid \boldsymbol{\sigma}_{h|K} \in \mathbb{P}_1,\ \forall K \in \mathcal{T}_h\right\} \subset S_1^{d\times d}, \tag{5.7}$$

where

$$T = \left\{\sigma \in L^2(\Omega) | \mathbf{u}\cdot\nabla\sigma \in L^2(\Omega), \sigma_{12} = \sigma_{21}\right\}, \tag{5.8}$$

So, the Oldroyd-B model is approached by the following problem:

Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$,

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in \mathbf{V}_h^0 \times M_h \times \mathbf{T}_h \ \text{ such that} & \\ a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\nabla\cdot\boldsymbol{\sigma}_h + \mathbf{f}, \mathbf{v}_h), & \forall \mathbf{v_h} \in \mathbf{V}_h^0 \\ b(\mathbf{u_h}, q_h) = 0, & \forall q_h \in M_h \\ (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathbf{u}_h\cdot\nabla\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\phi^*_{\partial K}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h) = (\mathbf{h}(\nabla\mathbf{u}, \boldsymbol{\sigma}), \boldsymbol{\tau}_h)\,, & \forall \boldsymbol{\tau}_h \in \mathbf{W} \end{cases} \tag{5.9}$$

with the interface and boundary adjoint-fluxes

$$\phi^{*,i}(\boldsymbol{\sigma}_h)|_{\partial K} = \left(\alpha|\mathbf{u}.\mathbf{n}_K| - \frac{1}{2}\mathbf{u}.\mathbf{n}_K\right)[[\boldsymbol{\sigma}_h]]_{\partial K}, \tag{5.10}$$

$$\phi^{*,\partial}(\boldsymbol{\sigma}_h) = -|\mathbf{u}.\mathbf{n}|\boldsymbol{\sigma}_h\chi_{\partial\Omega^-} \tag{5.11}$$

where $\alpha > 0$ is a parameter and $\chi_{\partial\Omega^-}$ denotes the characteristic function of $\partial\Omega^-$. Here $\mathbf{h}(\nabla\mathbf{u}, \boldsymbol{\sigma}) = 2\mu_e\mathbf{D}(\mathbf{u}) + \lambda_1\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right]$. The non-dimensional approah problem can be written as follows:

Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, find $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in \mathbf{V}_h^0 \times M_h \times \mathbf{T}_h$ such that

$$\begin{cases} \int_\Omega Re\left[(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h\cdot\mathbf{v}_h\right] + \int_\Omega(1-\lambda)\nabla\mathbf{u}_h : \nabla\mathbf{v}_h + \int_\Omega\boldsymbol{\sigma}_h : \nabla\mathbf{v}_h - \int_\Omega p_h\nabla\cdot\mathbf{v}_h = \int_\Omega\mathbf{f}\cdot\mathbf{v}_h \\ \int_\Omega q_h\nabla\cdot\mathbf{u}_h = 0 \\ \int_\Omega\boldsymbol{\sigma}_h:\boldsymbol{\tau}_h + \int_\Omega We\left[(\mathbf{u}_h\cdot\nabla)\boldsymbol{\sigma}_h\right]:\boldsymbol{\tau}_h - \int_\Omega We\left[(\nabla\mathbf{u}_h)\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h(\nabla\mathbf{u}_h)^t\right]:\boldsymbol{\tau}_h + (\phi^*_{\partial K}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h) = \int_\Omega 2\lambda\mathbf{D}(\mathbf{u}_h):\boldsymbol{\tau}_h \end{cases} \tag{5.12}$$

$\forall\mathbf{v}_h \in \mathbf{V}_h^0$, $\forall q_h \in M_h$, $\forall\boldsymbol{\tau}_h \in \mathbf{T}_h$.

From theorem (5.3.1), spaces $\mathbf{V}_h^0$ and M_h verified the uniform LBB-condition:

There exists $\beta > 0$ (independent of h) such that

$$\inf_{q\in M_h}\sup_{\mathbf{v}\in\mathbf{V}_h^0}\frac{\int_\Omega q\nabla\cdot\mathbf{v}}{\|q\|\cdot\|\nabla\mathbf{v}\|} \geqslant \beta > 0. \tag{5.13}$$

5.5 Algorithm to Solve the Discrete Oldroyd-B Problem

The discrete problem (5.12) leads to the solution of a highly nonlinear system of coupled equations. To solve the system of equations, the whole set of variables $\mathbf{u}$, p and $\boldsymbol{\sigma}$ using a technique requires excessive computer resources. So, to solve this elliptic-hyperbolic system we apply the decoupled technique. The extra-stress tensor is computing separately from the kinematic equations. From a fixed value for the velocity (and pressure) the extra-stress tensor is evaluated solving the tensorial transport equation (the third equation of (5.12)) by application of a fixed point method. Then the velocity field and pressure are updated with the current extra-stress tensor whose components are treated as known body forces, solving the resulting Navier-Stokes equation by the Newton-Raphson method. This procedure is iterated.

- Given $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$ the approach solution of iteration n, find $\left(\mathbf{u}_h^{n+1}, p_h^{n+1}\right)$, the solution of

$$\int_\Omega Re\left[(\mathbf{u}_h^{n+1}\cdot\nabla)\mathbf{u}_h^{n+1}\cdot\mathbf{v}_h\right] + \int_\Omega (1-\lambda)\nabla\mathbf{u}_h^{n+1} : \nabla\mathbf{v}_h - \int_\Omega p_h^{n+1}\nabla\cdot\mathbf{v}_h$$
$$= \int_\Omega \boldsymbol{\sigma}_h^n : \nabla\mathbf{v}_h + \int_\Omega \mathbf{f}\cdot\mathbf{v}_h$$

- Given $\mathbf{u}_h^{n+1}$ and $\boldsymbol{\sigma}_h^n = \boldsymbol{\sigma}_h^{n_0}$, find the solution $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_h^{n_{k+1}}$ of

$$\int_\Omega \boldsymbol{\sigma}_h^{n_{k+1}} : \boldsymbol{\tau}_h + We\int_\Omega \left[(\mathbf{u}_h^{n+1}\cdot\nabla)\boldsymbol{\sigma}_h^{n_{k+1}}\right] : \boldsymbol{\tau}_h + \left(\phi^*_{\partial K}(\boldsymbol{\sigma}_h^{n_{k+1}}), \boldsymbol{\tau}_h\right)$$
$$= \int_\Omega We\left[\boldsymbol{\sigma}_h^{n_k}\nabla\mathbf{u}_h^{k+1} + \nabla\mathbf{u}_h^{k+1}\boldsymbol{\sigma}_h^{n_k}\right] : \boldsymbol{\tau}_h + \int_\Omega 2\lambda\mathbf{D}(\mathbf{u}_h^{k+1}) : \boldsymbol{\tau}_h, k \geq 0$$

5.6 Numerical Results

This section is concerned with the application of the finite element method to obtain the numerical results for non-Newtonian viscoelastic Oldroyds-B fluid flows.
By the implementation of the finite element method in our own script in FreeFem++, we obtain the numerical solutions of the Oldroyd-B problem.

5.6.1 Validation of the code

To validate our code we consider the Oldroyd-B flow between two rigid walls where the flow is driven by a pressure difference along x-direction. This flow is laminar and referred as Poiseuille flows.
The velocity is uniaxial and has a parabolic profile and we suppose that $\dfrac{\partial p}{\partial x} = 1$. The analytic solution for the kinematic is given by [46, 31]

$$\begin{aligned} u_1(x,y) &= y\ (1-y) \ \ \text{(means } \mu = 0.125 Pa\, s) \\ u_2(x,y) &= 0 \\ p(x,y) &= x + C \text{ (C is a constant)} \end{aligned}$$

Substituting the velocity in the transport equations, we obtain by simple calculations, the components of the tensor as the functions of u_1 which can be written as

$$
\begin{aligned}
\sigma_{11} &= 2\lambda We \left(\frac{\partial u_1}{\partial y}\right)^2 \\
\sigma_{12} &= \lambda \left(\frac{\partial u_1}{\partial y}\right) \\
\sigma_{22} &= 0
\end{aligned}
\qquad (5.14)
$$

We consider the fluid is confined into a domain $\Omega = [0, 10] \times [0, 1]$. The no-slip conditions on the two rigid walls are given by $u_1 = 0$, $u_2 = 0$. We assume that $u_2 = 0$ everywhere at the inlet and $u_1(x, y) = y\ (1 - y)$ as the exact solution. At outlet we impose $u_2 = 0$. The condition for stress tensor on the upstream boundary section $\partial\Omega^-$ agree with the exact solution (5.14).

The problem has been solved using four grids obtained by successive refinements dividing each triangle into four new triangles starting with a coarse mesh with 344 elements.

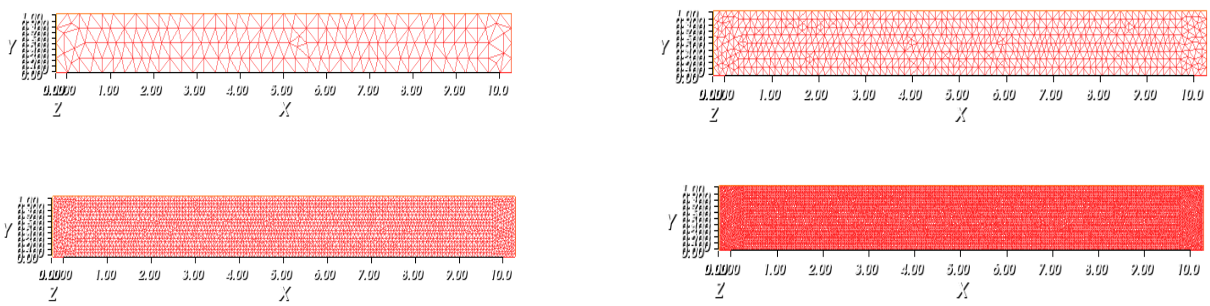


Figure 5.1: *Different meshes used. Fom the left to right and top to botton: mesh with 344 elements, 1374 elements, 5410 elements, 22654 elements, respectively.*

We consider the problem with $Re = 1$, $We = 1$ and $\lambda = 0.1$.

The Table 5.1 characterizes the mesh through the diameter h, number of elements, degree of freedoms.

Grid	h	No. of elements	$\mathbb{P}_2$ nodes	$\mathbb{P}_1$ nodes	$\mathbb{P}_1dc$ nodes
Grid1	0.372678	344	777	217	1032
Grid2	0.18815	1374	2925	776	4122
Grid3	0.10233	5410	11173	2882	16230
Grid4	0.0511443	22654	46013	11680	67962

Table 5.1: Characterizations of the grids

In each case, we evaluate the error of fluid velocity in $\mathbf{H}^1$-norm and the error of the components of the tensor in L^2-norm which are respectively defined by

$$err_u = \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)}$$

and

$$err_{\sigma_{ij}} = \|\sigma_{ij} - \sigma_{h,ij}\|_{L^2(\Omega)},\ \ i,j = 1,2.$$

The results obtained for $\mathbf{u}$ and $\sigma_{ij}, i,j = 1,2$ over the different meshes are present in the following table (Table 5.2).

Error	Grid1	Grid2	Grid3	Grid4	Slope of the log-log plot
err_u	0.0016779	0.00065528	0.00034032	0.00017678	1.1279
$err_{\sigma_{11}}$	0.025702	0.00615	0.0015003	0.0003415	2.18775
$err_{\sigma_{12}}$	7.8211×10^{-5}	3.4432×10^{-5}	1.7488×10^{-5}	8.5971×10^{-6}	1.1117
$err_{\sigma_{22}}$	4.7276×10^{-5}	2.7917×10^{-5}	1.0495×10^{-5}	5.3389×10^{-6}	1.142

Table 5.2: Error of the velocity field and tensor components

The good convergence of results for all kinematic can be confirmed by the slope value, which gives us the rate of convergence to the exact solution. We used the least squares approximation to find the slope of the log-log plot of the error of the velocity and the tensor components. The following plots show us the error curves:

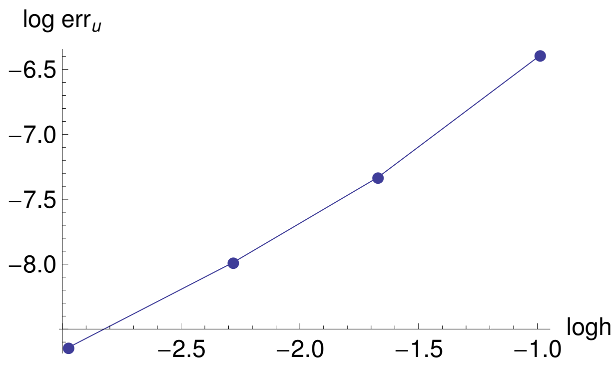


Figure 5.2: *Log-log plot of the error of the velocity.*

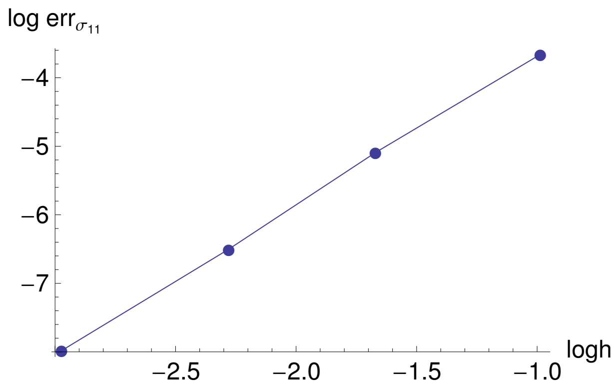


Figure 5.3: *Log-log plot of the error of the component σ_{11} of the tensor.*

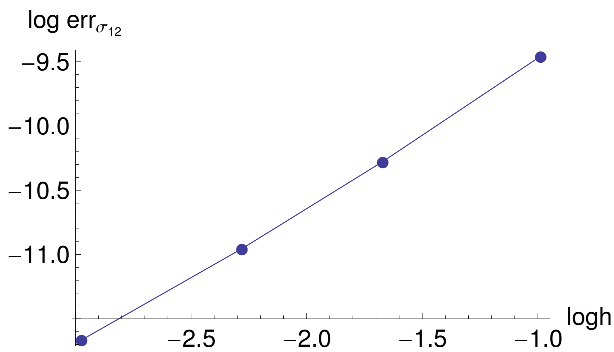


Figure 5.4: *Log-log plot of the error of the component* σ_{12} *of the tensor.*

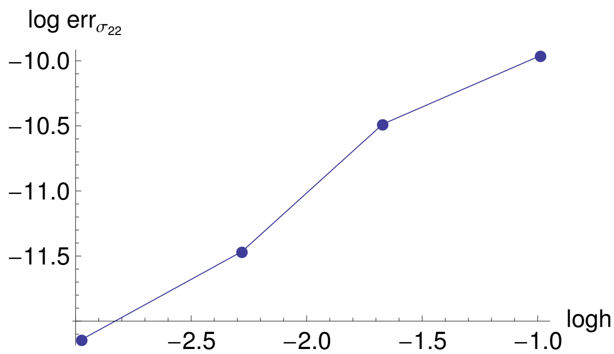
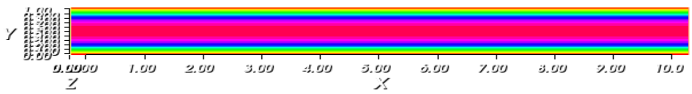


Figure 5.5: *Log-log plot of the error of the component* σ_{22} *of the tensor.*

The values for the rate of convergence (the slope) which we obtained guarantees the errors for all the variables approaches to zero as h tends to zero, which we expected theoretically. We are in conditions to affirm that the numerical solution converges to the exact solution. So, the algorithm is convergent and our code runs well.

The approach solution obtained with 22654 elements is illustrated graphically in the next following figures:



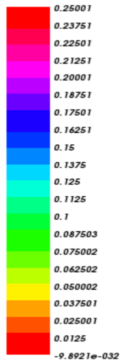
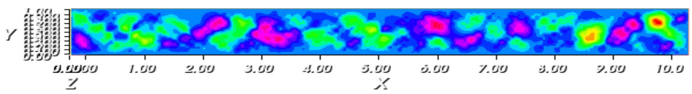


Figure 5.6: *Contours of the first component of the velocity.*



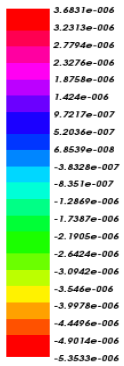
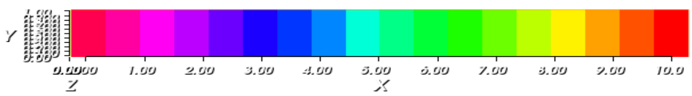


Figure 5.7: *Contours of the second component of the velocity.*



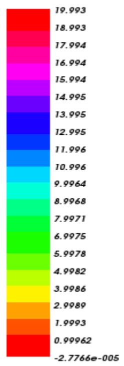
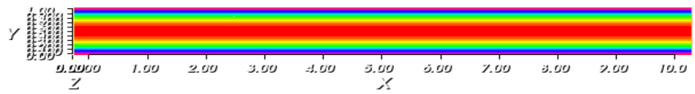


Figure 5.8: *Contours of the pressure.*



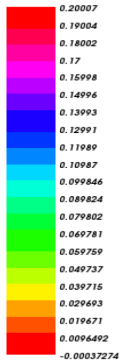
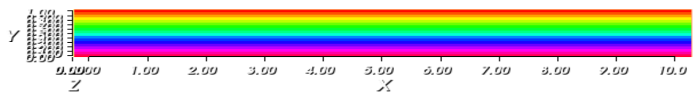


Figure 5.9: *Contours of the first component of the tensor.*



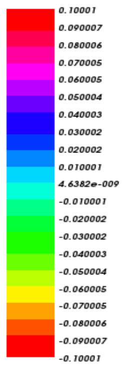
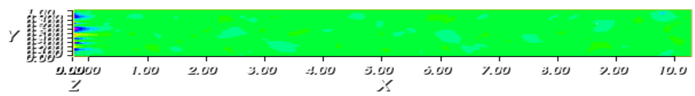


Figure 5.10: *Contours of the second component of the tensor.*



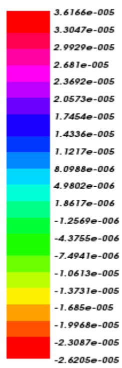


Figure 5.11: *Contours of the third component of the tensor.*

5.6.2 Results for a four-to-one abrupt contraction

In this subsection, we consider that the fluid flows in an abrupt contraction (or $4:1$ planar contraction) subjected to suitable boundary conditions. This type of flow is interesting theoretically and practically and has been studied by many authors [43], since 1988. These problems have a lot of applications in polymer processing, especially in extrusion and injection moulding.

We consider that the fluid is confined into a domain Ω with its boundary $\partial\Omega = \bigcup_{k=1}^{8} \partial\Omega_k$ which is shown in figure 5.12.

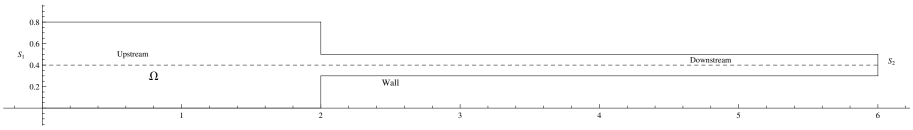


Figure 5.12: *Computational domain* Ω *for a* $4:1$ *abrupt contraction.*

This domain Ω consists of six boundaries as rigid wall denoted by $\partial\Omega_w = \bigcup \partial\Omega_k$, with k=$1, 2, 3, 5, 6, 7$, an inlet or inflow boundary at upstream section $\partial\Omega_8 = S_1$, and an outlet or outflow boundary at downstream section $\partial\Omega_4 = S_2$. The fluid enters into the domain through the upstream section S_1. We consider an inflow parabolic profile at the upstream section for the velocity field and homogeneous no-slip conditions on the wall. To obtain the Poiseuille velocity profiles before the contraction and at downstream, the computational domain Ω is assumed to be long enough at upstream and downstream sections. In our referential domain, we consider the length of upstream section is $5r_1 = 2$, with r_1 the radius of upstream, while the length of downstream section is taken as $10r_1 = 4$. The respective widths of upstream and downstream sections are $2r_1 = 0.8$ and $2r_2 = 0.2$. Taking into account the boundary conditions of Saramito [43], we impose the boundary conditions, for the problem, which is adapted to our computational domain Ω.

We impose the following boundary conditions:

- Inflow boundary conditions for the velocity and the stresses at the upstream section i.e., on $\partial\Omega_8 = S_1$:

$$
\begin{aligned}
u_1 &= \frac{r_2}{8r_1^2} y(2r_1 - y) = 0.078125y\ (0.8 - y) \\
u_2 &= 0 \\
\sigma_{11} &= 2\lambda We \left(\frac{\partial u_1}{\partial y}\right)^2 \\
\sigma_{12} &= \lambda \left(\frac{\partial u_1}{\partial y}\right) \\
\sigma_{22} &= 0
\end{aligned}
$$

- Outflow boundary conditions at downstream section i.e., on $\partial\Omega_4 = S_2$:

$$u_1 = 0$$

$$u_2 = 0$$

- The boundary conditions for the velocity at the wall i.e., on $\partial\Omega_w$:

$$u_2 = 0$$

So, the Oldroyd-B model problem to compute the flow in an abrupt contraction can be written as

$$
\begin{cases}
Re\,[(\mathbf{u}\cdot\nabla)\mathbf{u}] - (1-\lambda)\Delta\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}, \quad \text{in} \ \ \Omega \\
\nabla\cdot\mathbf{u} = 0, \quad \text{in} \ \ \Omega \\
\boldsymbol{\sigma} + We\,[(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}] = 2\lambda\mathbf{D}(\mathbf{u}) + We\left[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t\right], \quad \text{in} \ \ \Omega \\
u_1 = 0.078125y(0.8-y), \quad \text{on} \ \ \partial\Omega_8 = S_1 \\
u_2 = 0, \quad \text{on} \ \ \partial\Omega_4 = S_2 \\
u_1 = 0, u_2 = 0, \quad \text{on} \ \ \partial\Omega_w \\
\sigma_{11} = 2\lambda We\left(\frac{\partial u_1}{\partial y}\right)^2, \quad \text{on } S_1 \\
\sigma_{12} = \lambda\left(\frac{\partial u_1}{\partial y}\right), \quad \text{on } S_1 \\
\sigma_{22} = 0, \quad \text{on } S_1
\end{cases}
\tag{5.15}
$$

For the numerical results, we use the exact solution of fully developed Poiseuille flow in straight pipe as the initial condition for the tensor while the solution of the Stokes problem is the initial condition for the velocity.

To obtain the numerical simulations of the flow, we take $\lambda = 0.1$ and we discretize the domain Ω on a mesh with 2066 elements where at 4405 $\mathbb{P}_2$ nodes for the velocity, 1170 $\mathbb{P}_1$ nodes for the pressure and 6198 $\mathbb{P}_1dc$ nodes for the tensor are defined (figure 5.13).

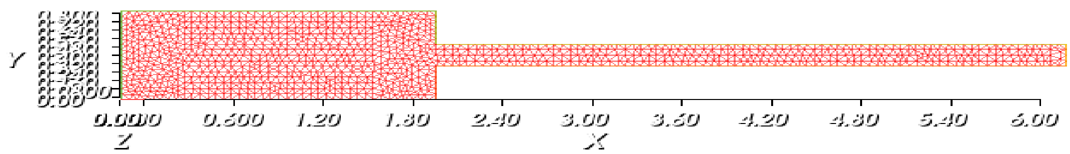


Figure 5.13: *Mesh with* 2066 *elements.*

Like we expected at upstream and at downstram sections, where the viscous forces are dominant, the flow is laminar. In the opening of narrower domain, we observed a different behavior, where inertial forces are fallen. For this reason, we decide to present a partial view of variables envolved in the problem in a neighborhood of contraction.

Newtonian flows

The simulations were performed to Newtonian flows. We fixed $We = 0$ and we take severals *Reynolds* numbers between 1 and 500 and compare the results. We observe that the quantitative behavior for the kinematic is almost same (Fig. 5.14 - 5.16), (Fig. 5.17 - 5.19), (Fig. 5.20 - 5.22) with a small increase of maximum values and small decrease of minimum values. The pressure is constant along the y-direction and varies linearly with x. In fact, the behavior of pressure is very similar that we observe for the Poiseuille flows. The qualitative behavior can be observed and compared easily through the stream function (Fig. 5.23 - 5.25). We observe that there is a recirculation vortex in the corner. These vortex are shrinking when the Reynolds increase. This is the effect of inertia which makes the fluid speed up to enter the narrow part of the domain (tube), pulling the fluid out of the corner to inside the tube and thus decreasing the vortex . To prove our description, of all tests performed we select 3 cases with $Re = 1, 100$ and 500 and we present the numerical results.

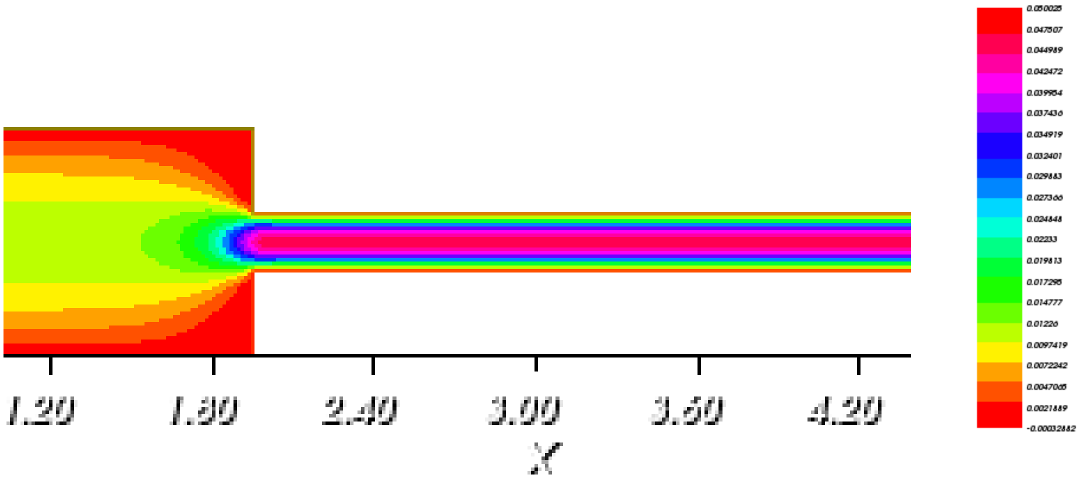


Figure 5.14: *The first component of the velocity (partial view) with* $Re = 1$.

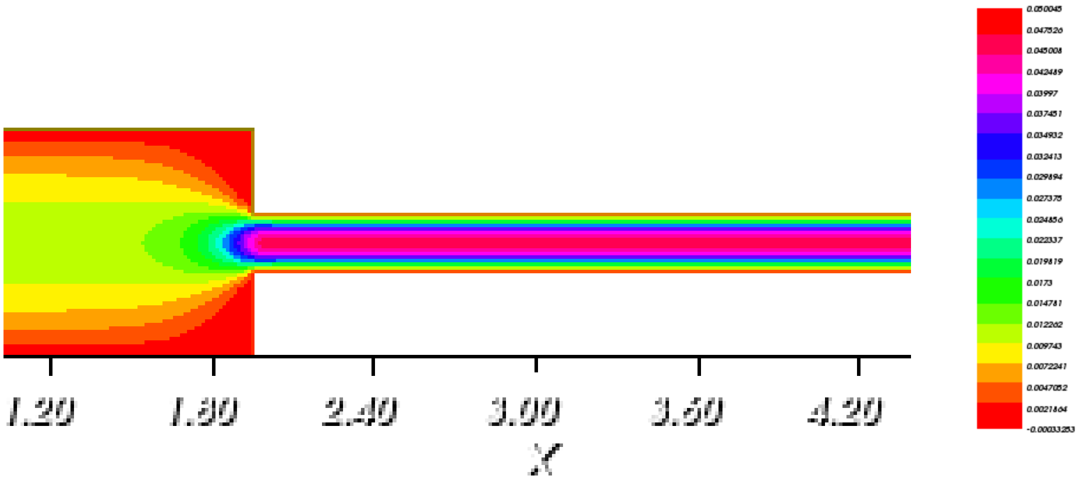


Figure 5.15: *The first component of the velocity (partial view) with* $Re = 100$.

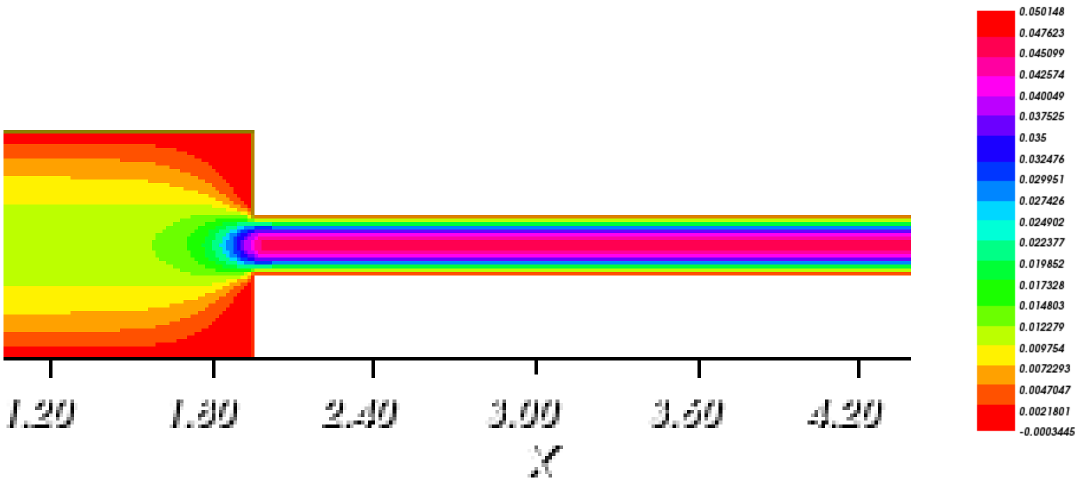


Figure 5.16: *The first component of the velocity (partial view) with* $Re = 500$.

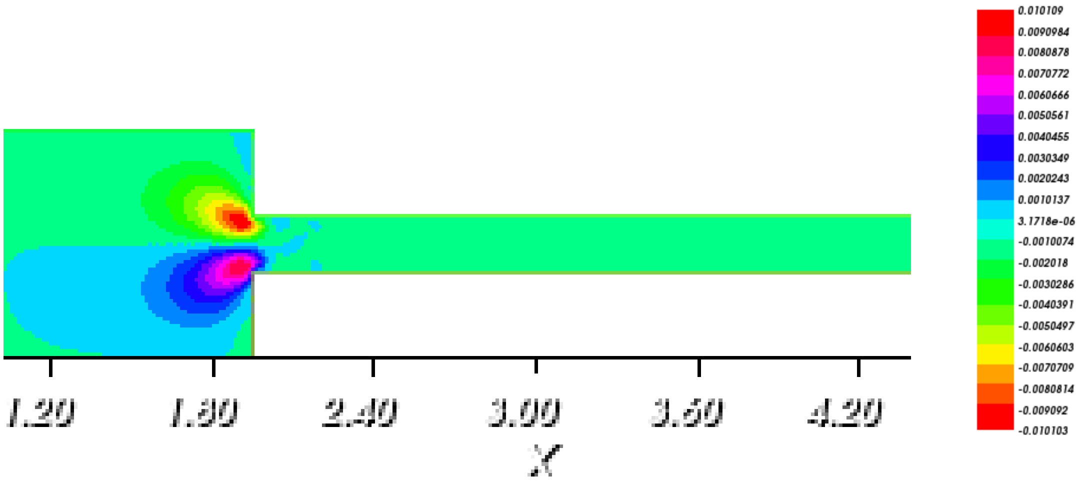


Figure 5.17: *The second component of the velocity (partial view) with* $Re = 1$.

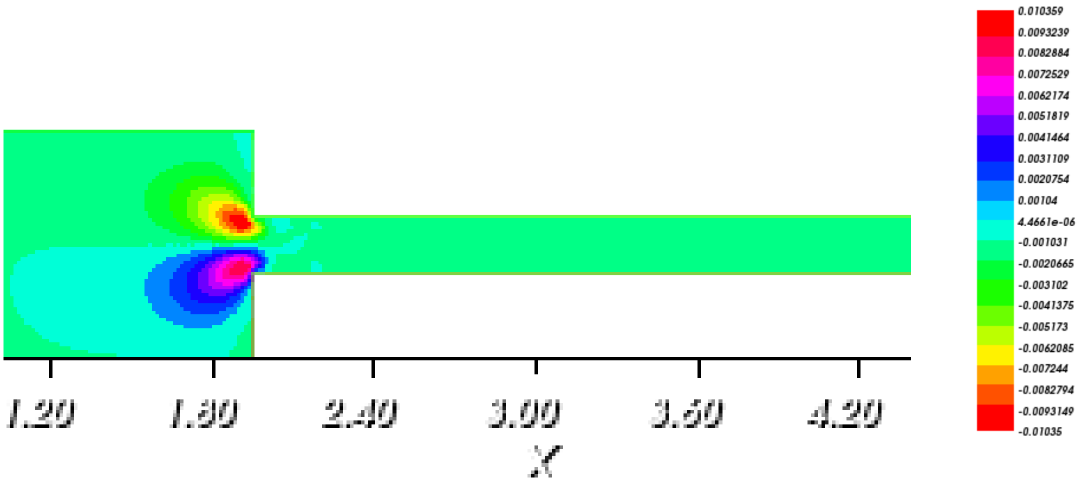


Figure 5.18: *The second component of the velocity (partial view) with* $Re = 100$.

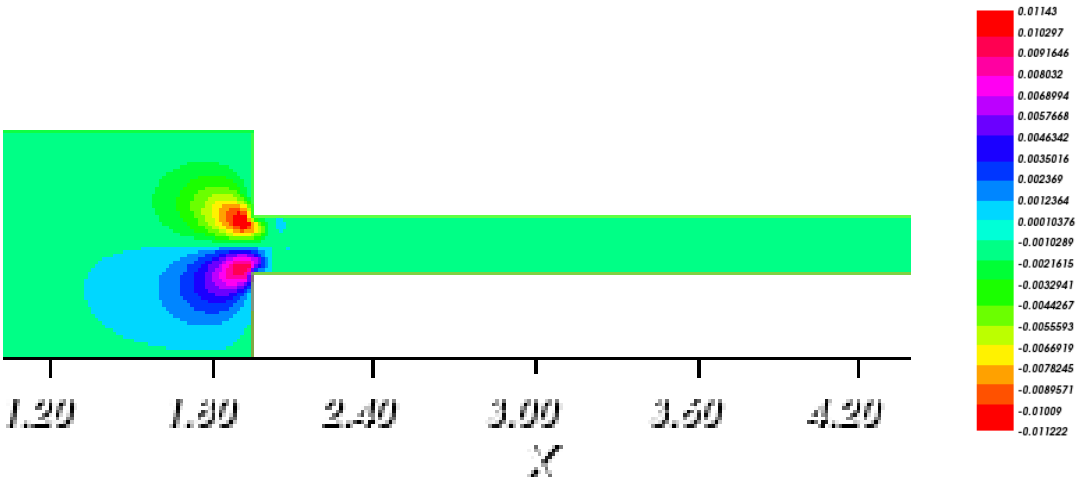


Figure 5.19: *The second component of the velocity (partial view) with* $Re = 500$.

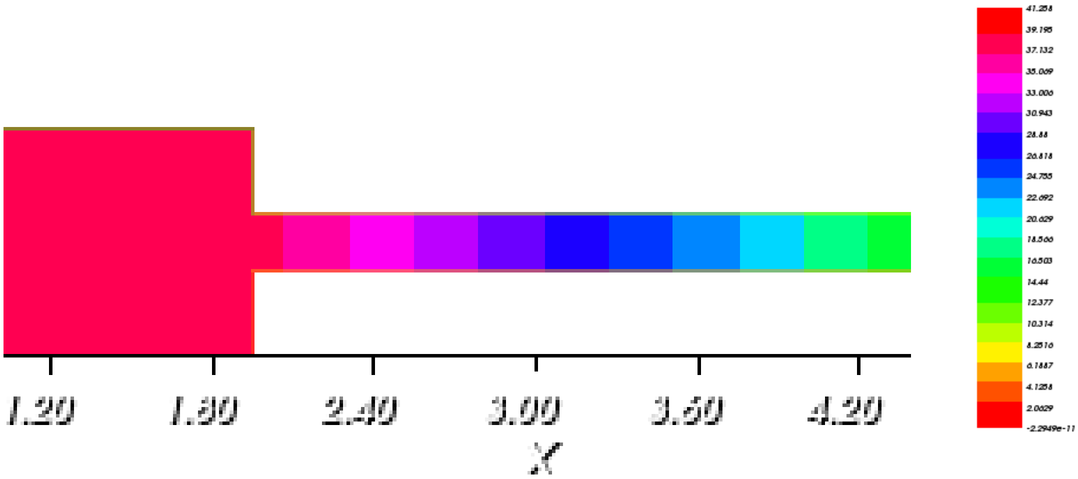


Figure 5.20: *The pressure (partial view) with Re* $= 1$.

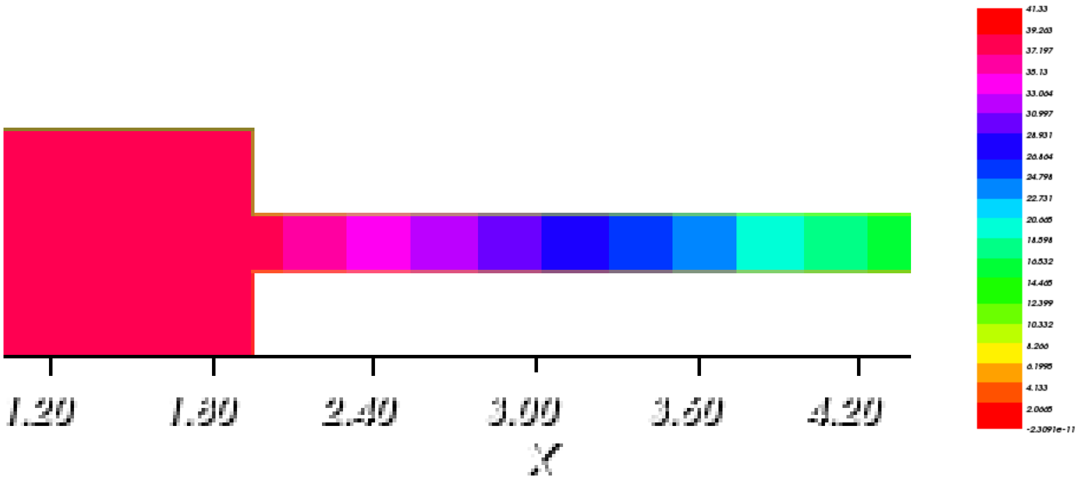


Figure 5.21: *The pressure (partial view) with Re* $= 100$.

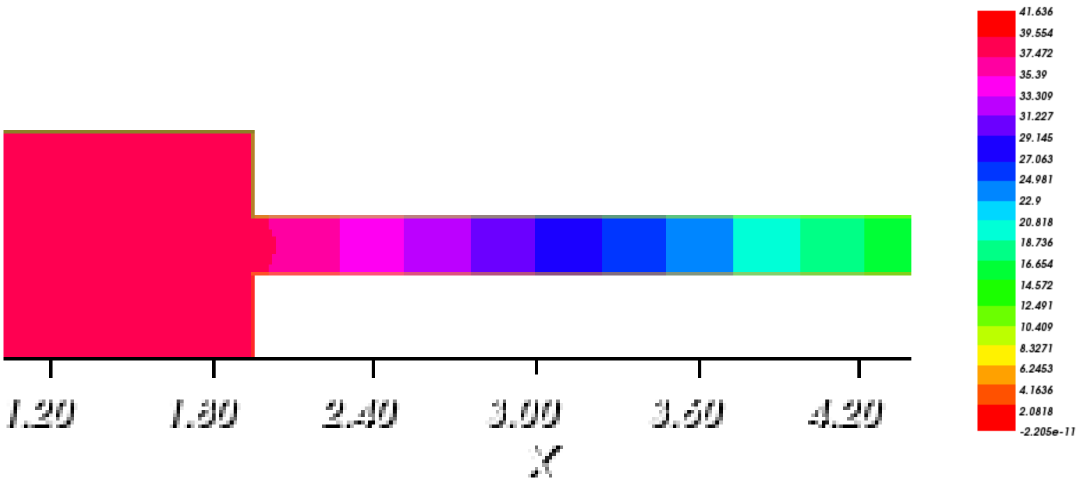


Figure 5.22: *The pressure of the velocity (partial view) with Re* $= 500$.

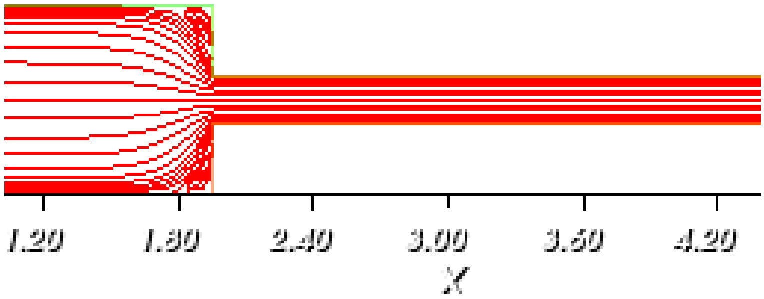


Figure 5.23: *The stream function (partial view) with* $Re = 1$.

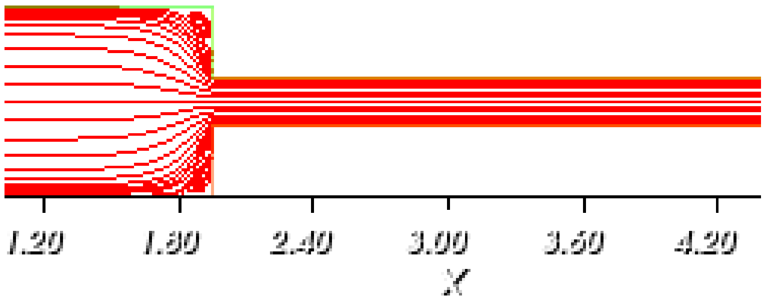


Figure 5.24: *The stream function (partial view) with* $Re = 100$.

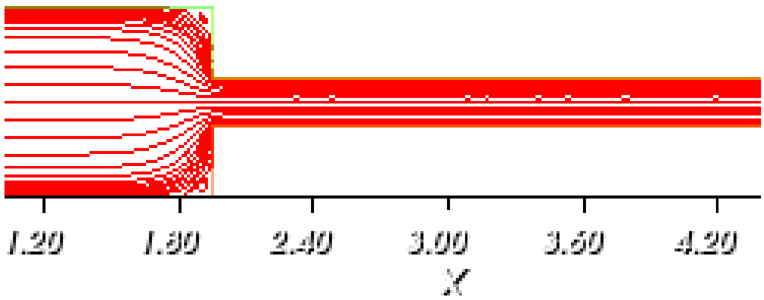


Figure 5.25: *The stream function (partial view) with* $Re = 500$.

Viscoelastic flows

The simulations were performed to viscoelastic flows. For all performed tests, we observed that the convergence of the algorithm is influenced by the values of Reynolds and Weissenberg, the latter being the main cause for the divergence of the algorithm. In fact, the two re-entrant corners are one of the reason for which the problem fails to converge if we consider the high value of Weissenberg number. This result may be directly related to the dissipative instability of the model in the elongation flow dominated in the contraction flow near the entrance region.

We tested various values of Reynolds to seek the limit of the Weissenberg value, for which the algorithm converges. We observe that when we increase the Re, threshold for We decreases. Table 5.3 shows the maximum values of Weissenberg numbers We for respective values of Reynold numbers Re, for which the results are convergent.

Re	1	50	100	250	500
We_{max}	5.13	5.08	5.03	4.86	4.60

Table 5.3: Maximum values of We (We_{max}) for respective values of Re.

For all performed tests, we analyzed the behavior of the velocity, pressure and tensor. We observed that the qualitative behaviors of the kinematic are the same for Newtonian flows although we can observe the differences of the types of recirculantions and their zones for the viscoelastic flows in relation to the Newtonian fluids flows through the stream function (compare the figures (5.23 - 5.25) with figures (5.26 - 5.33). The center of circulating zone moves from the re-entrant corner to the salient edge. We observed the elastic effects. The speeding up to along streamlines in the center causes an elastic tension along these streamlines which exerts a pull on the fluid directly upstream and pushes the fluid on the sides back into the corner and this causes a recirculation zone wich the intensity and the length increase as a combined effects between Re and We parameters.

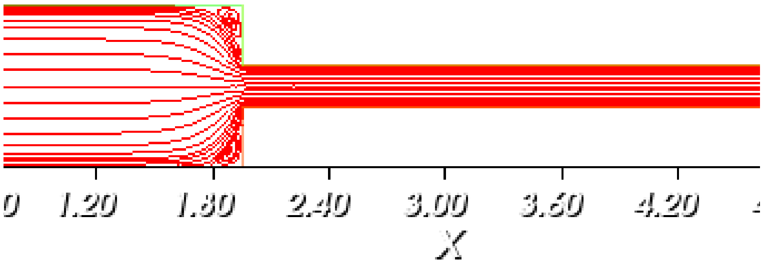


Figure 5.26: *The stream function (partial view) with $Re = 1$ and $We = 1$.*

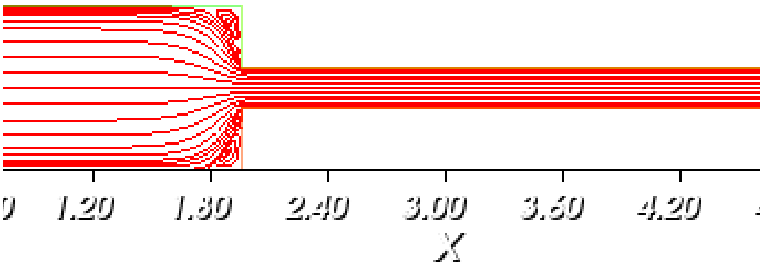


Figure 5.27: *The stream function (partial view) with $Re = 1$ and $We = 4.6$.*

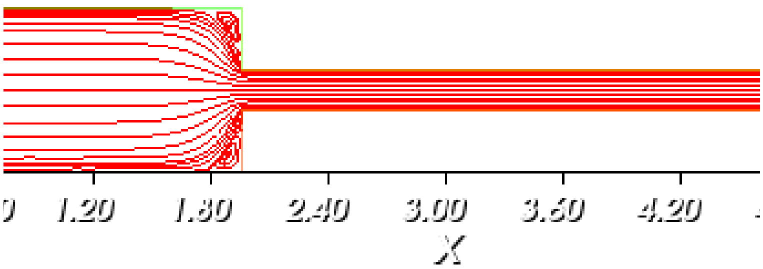


Figure 5.28: *The stream function (partial view) with $Re = 1$ and $We = 5.13$.*

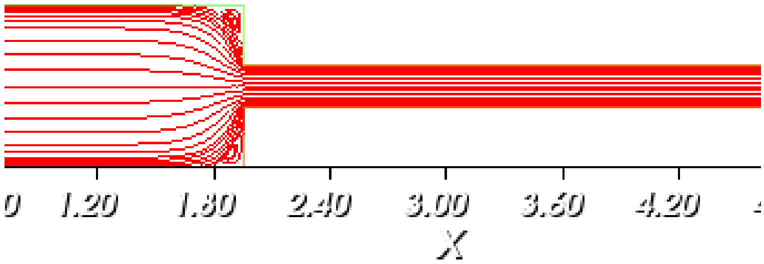


Figure 5.29: *The stream function (partial view) with $Re = 100$ and $We = 1$.*

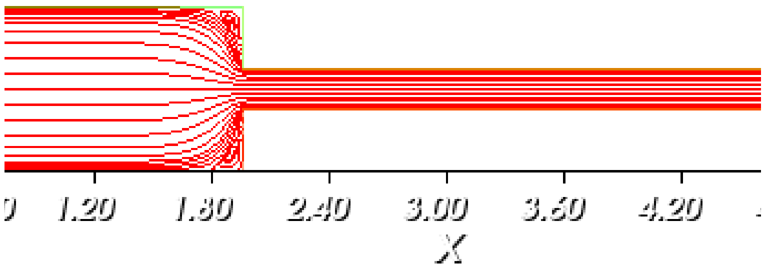


Figure 5.30: *The stream function (partial view) with $Re = 100$ and $We = 4.6$.*

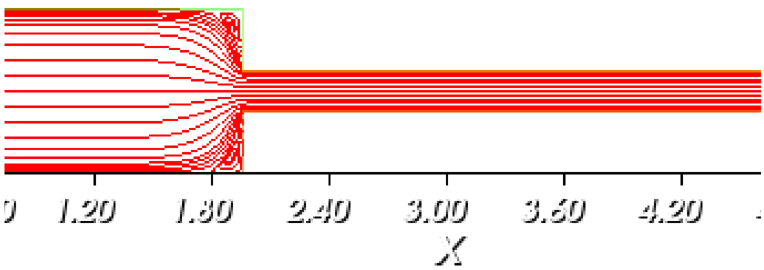


Figure 5.31: *The stream function (partial view) with $Re = 100$ and $We = 5.03$.*

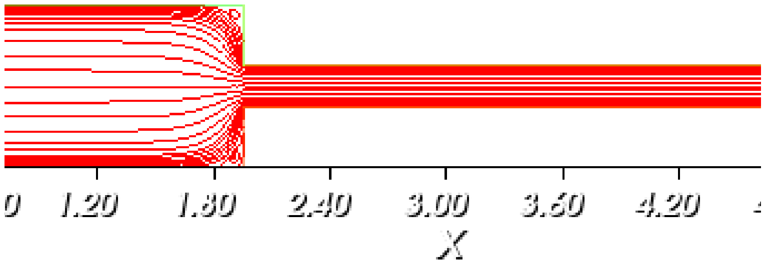


Figure 5.32: *The stream function (partial view) with Re* $= 500$ *and* $We = 1$.

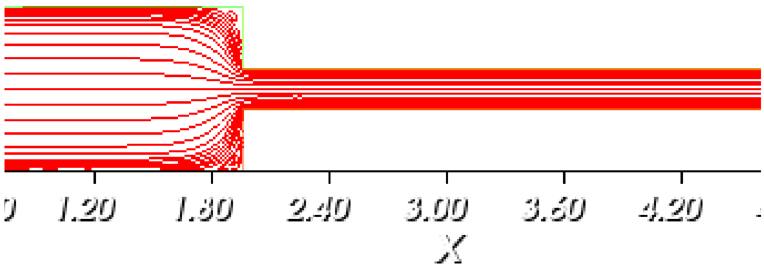


Figure 5.33: *The stream function (partial view) with Re* $= 500$ *and* $We = 4.6$.

We observe that the big differences of the behaviors of the components of tensors occur for σ_{11} and σ_{22}. For theses components, the influence of We is notable. For a fix value of Re, when the We increase we see the effects as to extend from the corners into the tube and in the case of σ_{22} also to behind the entry of this. The next figures show us the behavior of the components of tensor.

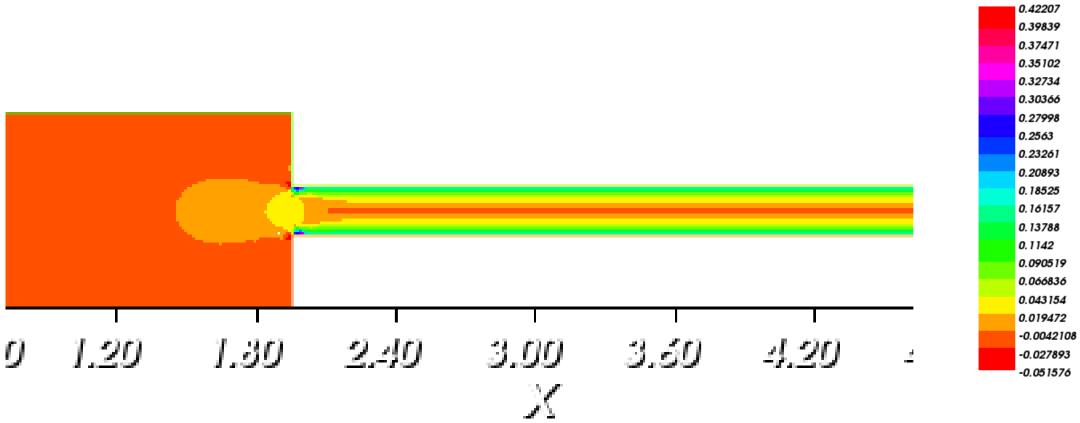


Figure 5.34: *The component of* σ_{11} *(partial view) with* $Re = 1$ *and* $We = 1$.

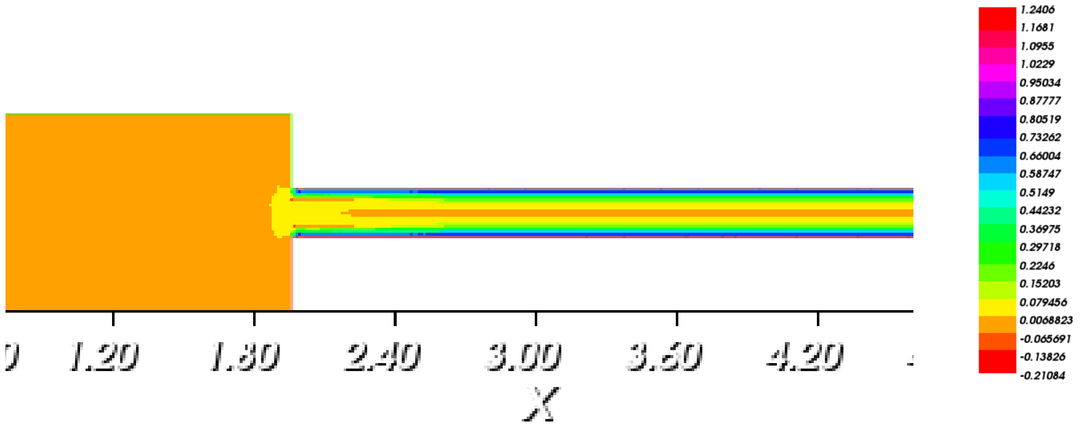


Figure 5.35: *The component of* σ_{11} *(partial view) with* $Re = 1$ *and* $We = 4.6$.

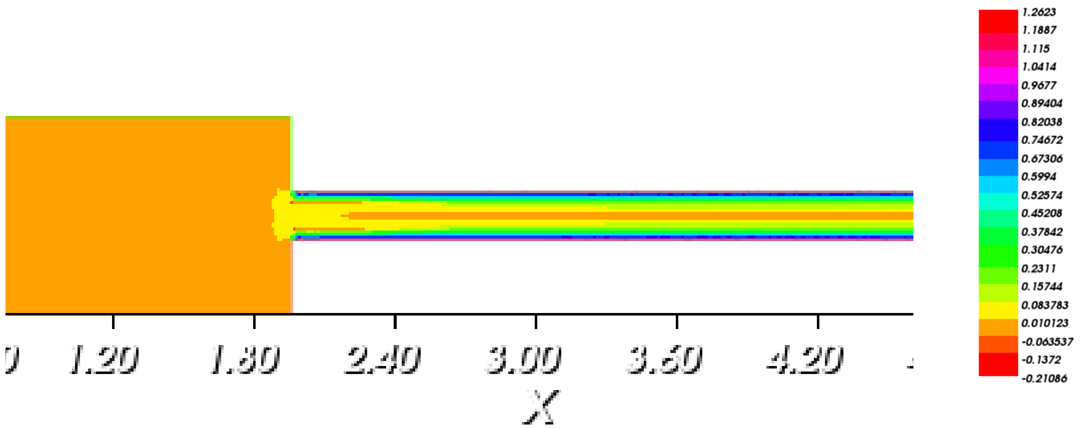


Figure 5.36: *The component of* σ_{11} *(partial view) with* $Re = 1$ *and* $We = 5.13$.

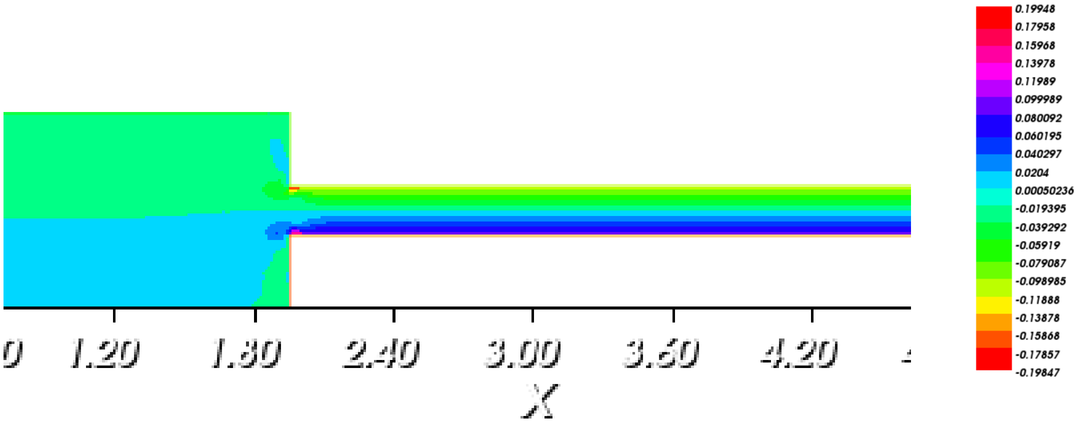


Figure 5.37: *The component of* σ_{12} *(partial view) with* $Re = 1$ *and* $We = 1$.

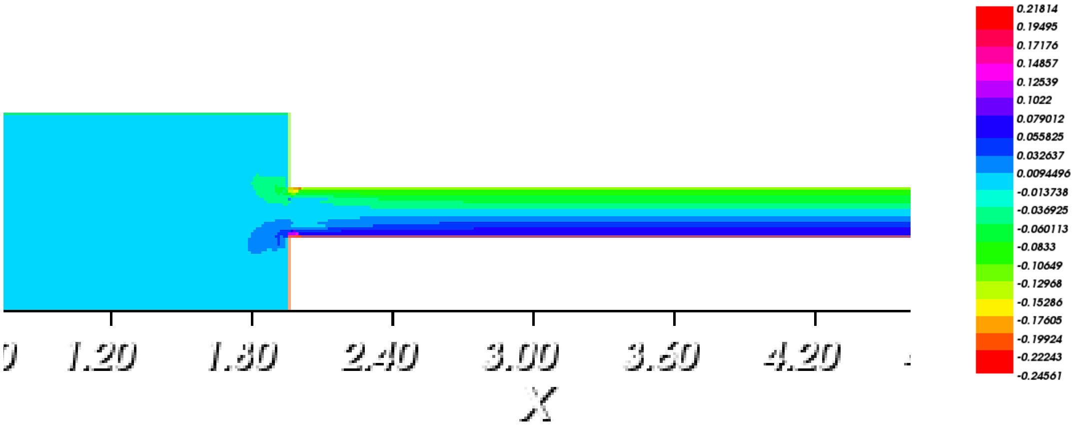


Figure 5.38: *The component of* σ_{12} *(partial view) with* $Re = 1$ *and* $We = 4.6$.

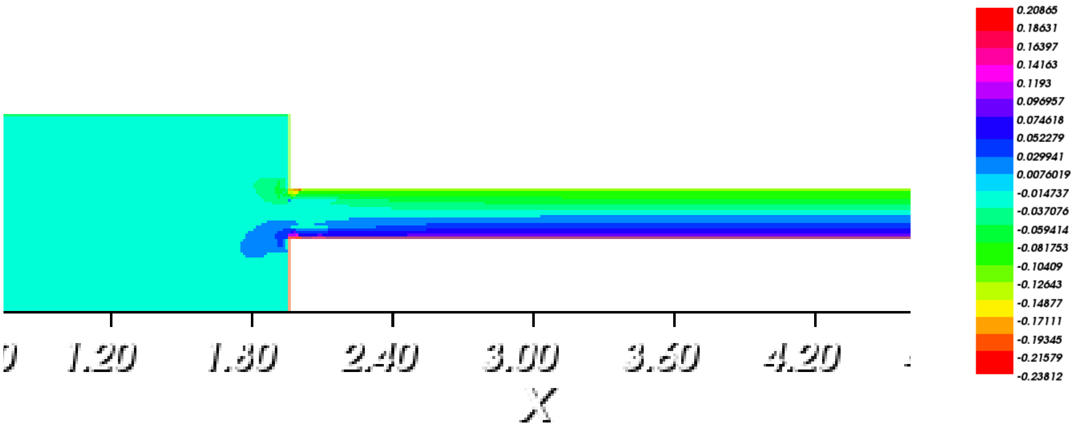


Figure 5.39: *The component of* σ_{12} *(partial view) with* $Re = 1$ *and* $We = 5.13$.

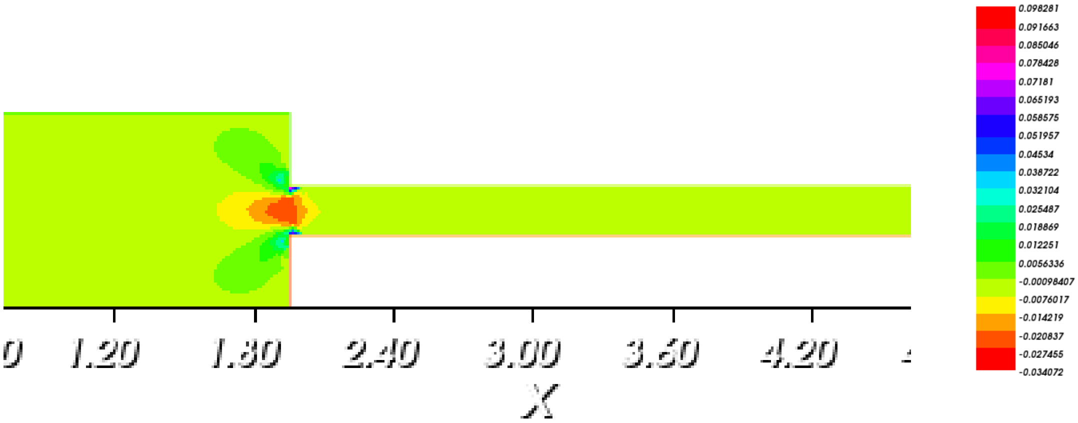


Figure 5.40: *The component of σ_{22} (partial view) with $Re = 1$ and $We = 1$.*

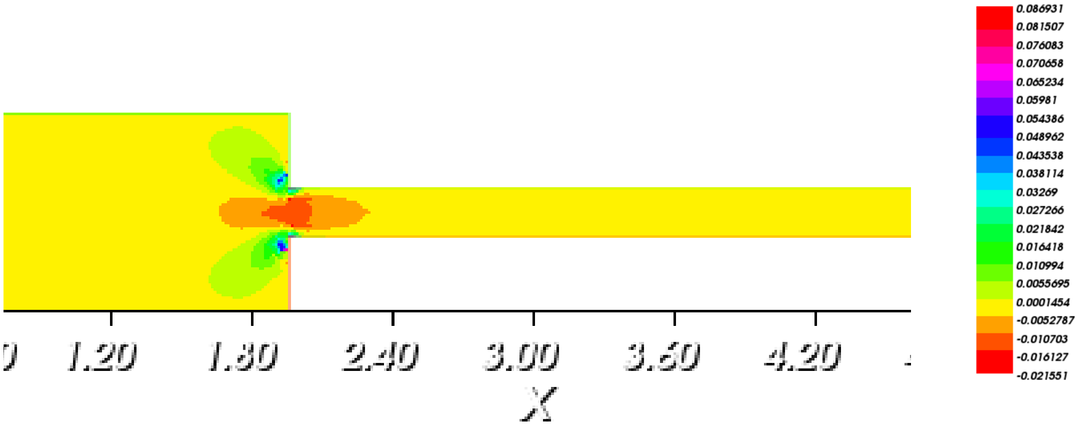


Figure 5.41: *The component of σ_{22} (partial view) with $Re = 1$ and $We = 4.6$.*

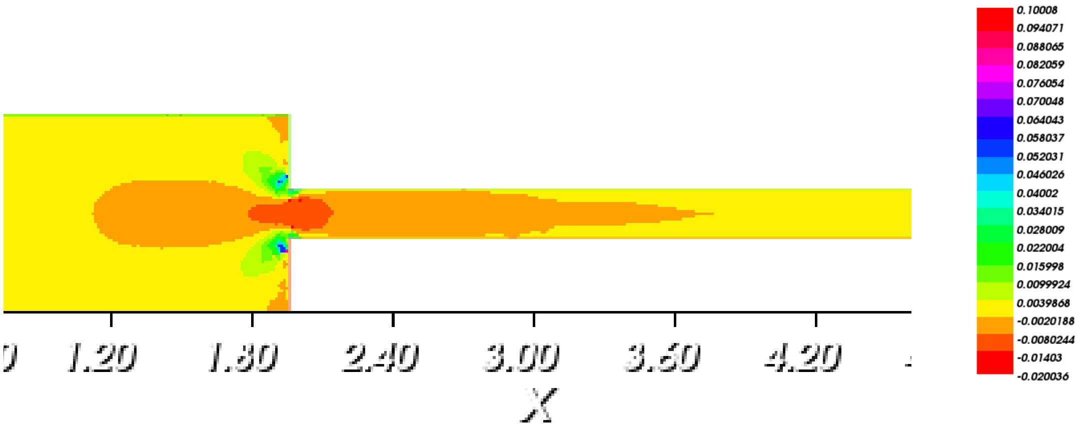


Figure 5.42: *The component of σ_{22} (partial view) with $Re = 1$ and $We = 5.13$.*

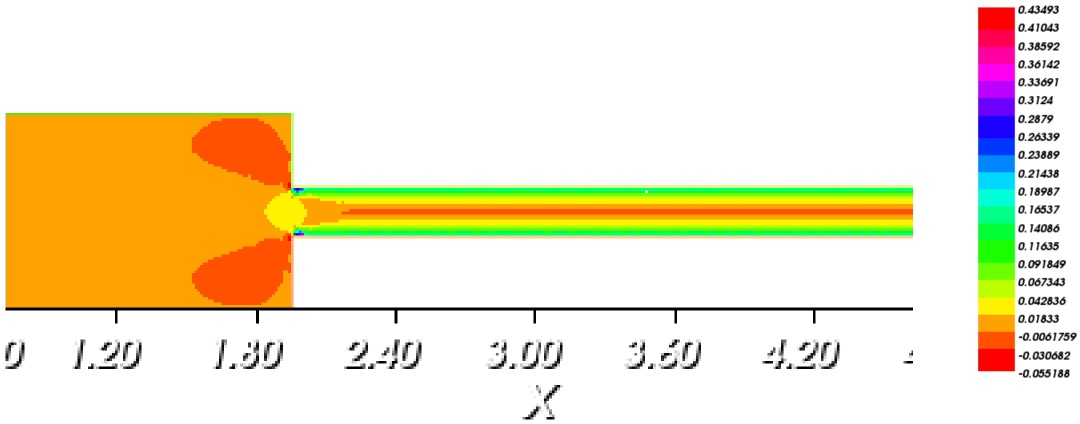


Figure 5.43: *The component of σ_{11} (partial view) with $Re = 100$ and $We = 1$.*

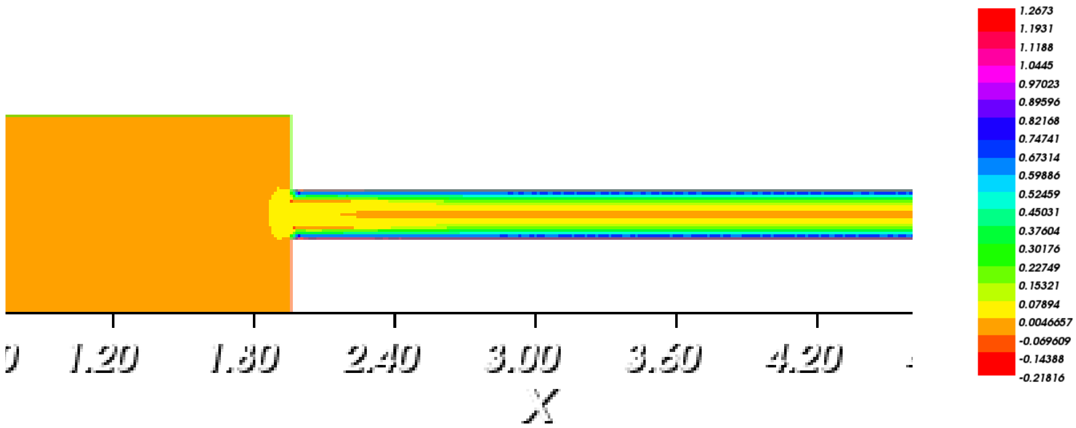


Figure 5.44: *The component of σ_{11} (partial view) with $Re = 100$ and $We = 4.6$.*

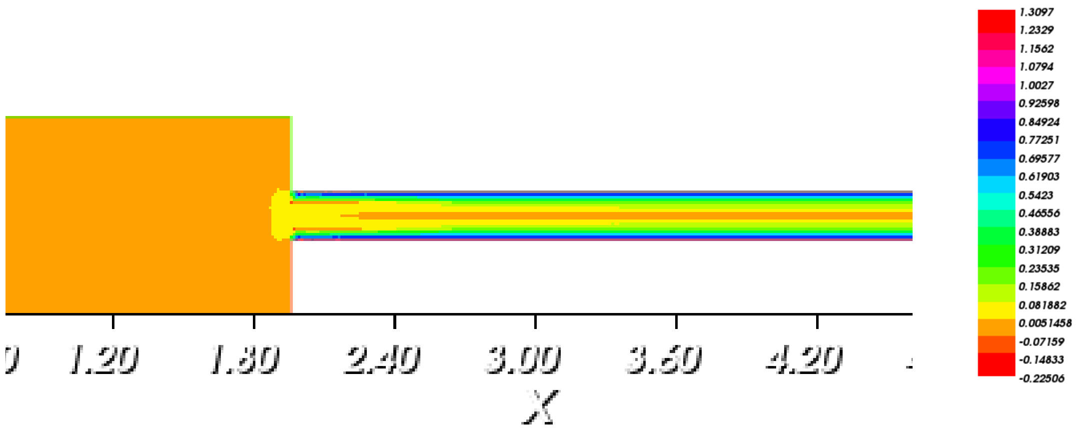


Figure 5.45: *The component of σ_{11} (partial view) with $Re = 100$ and $We = 5.03$.*

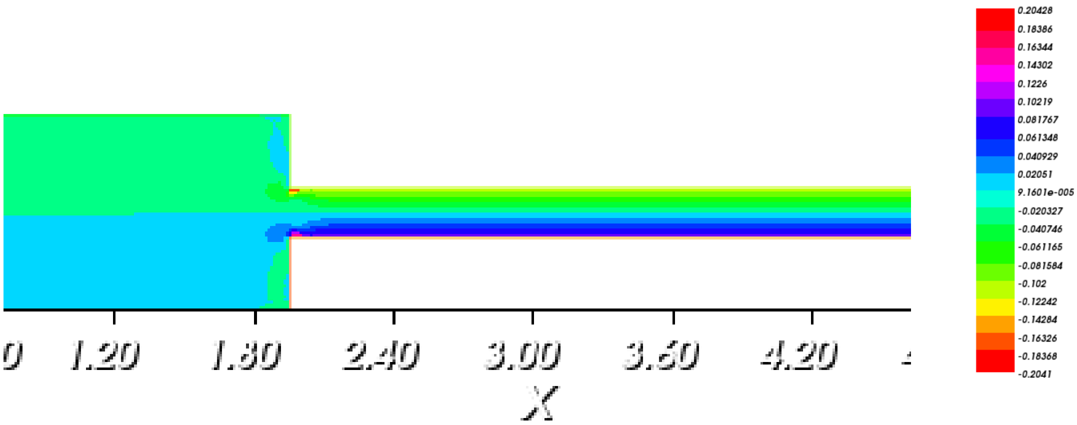


Figure 5.46: *The component of* σ_{12} *(partial view) with* $Re = 100$ *and* $We = 1$.

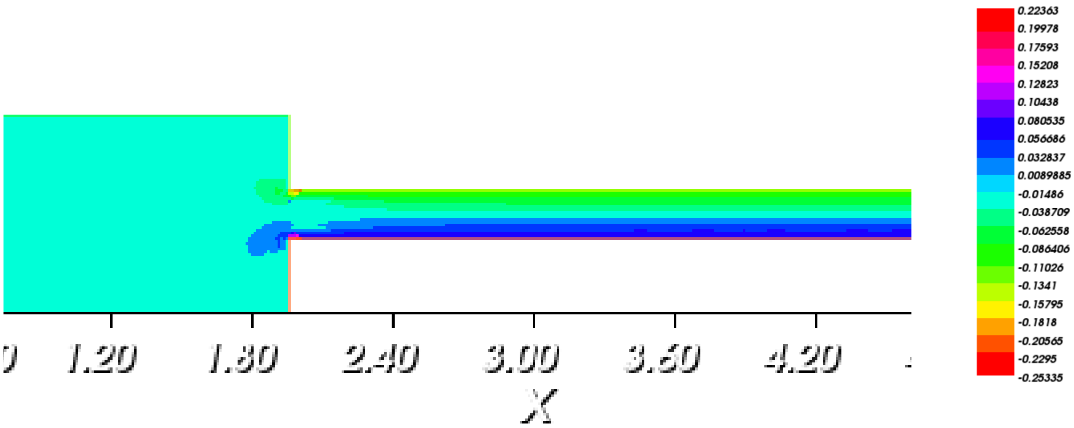


Figure 5.47: *The component of* σ_{12} *(partial view) with* $Re = 100$ *and* $We = 4.6$.

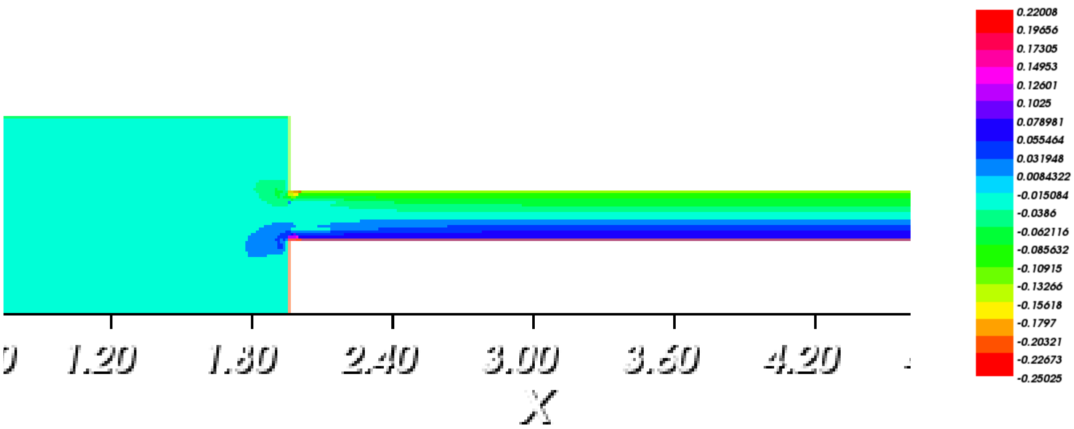


Figure 5.48: *The component of* σ_{12} *(partial view) with* $Re = 100$ *and* $We = 5.03$.

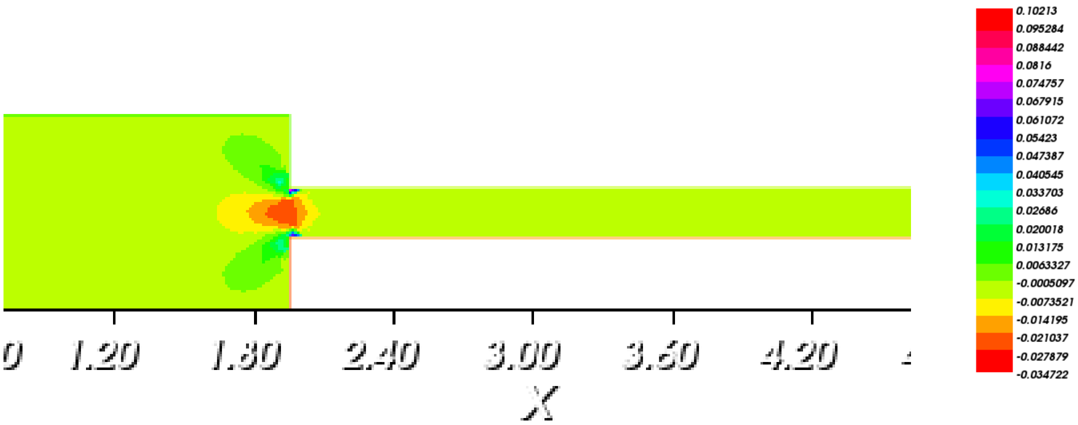


Figure 5.49: *The component of* σ_{22} *(partial view) with* $Re = 100$ *and* $We = 1$.

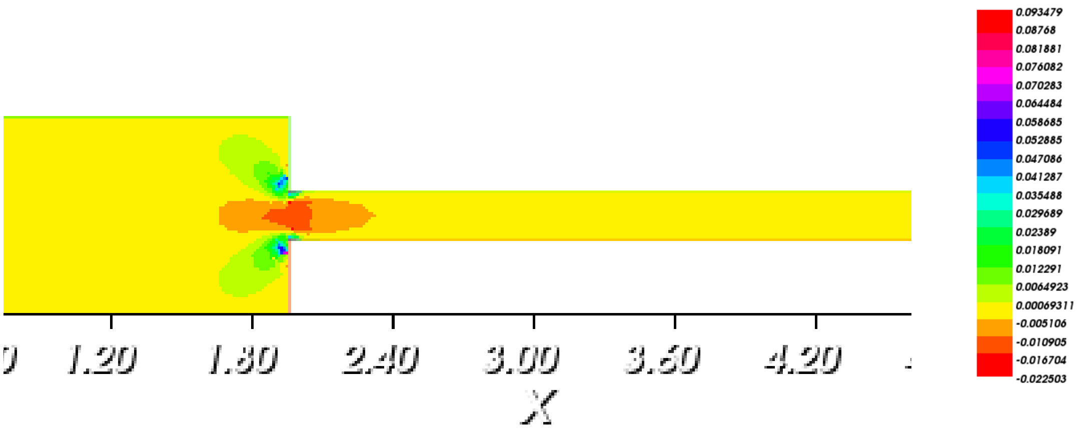


Figure 5.50: *The component of* σ_{22} *(partial view) with* $Re = 100$ *and* $We = 4.6$.

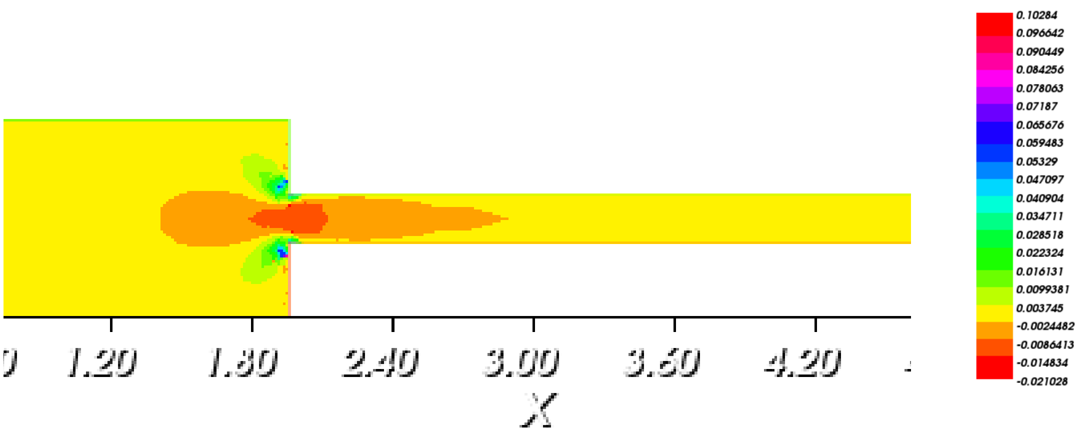


Figure 5.51: *The component of* σ_{22} *(partial view) with* $Re = 100$ *and* $We = 5.03$.

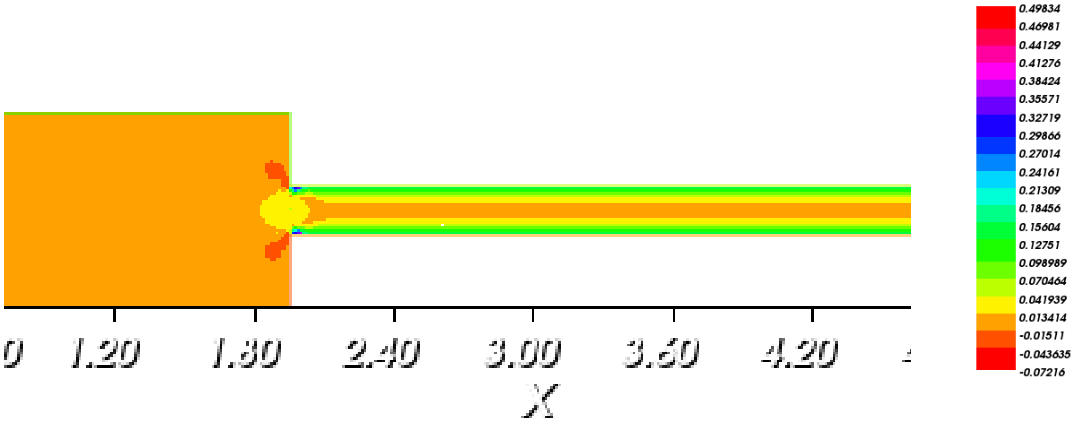


Figure 5.52: *The component of* σ_{11} *(partial view) with* $Re = 500$ *and* $We = 1$.

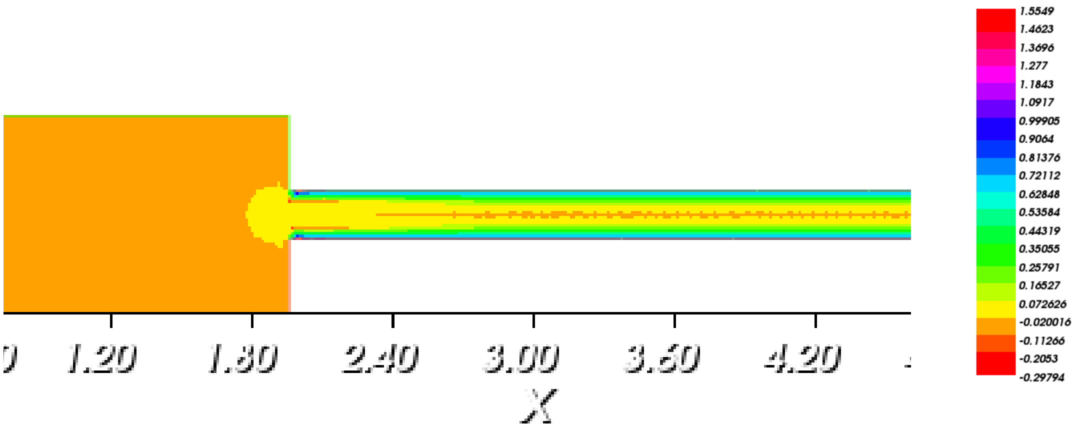


Figure 5.53: *The component of* σ_{11} *(partial view) with* $Re = 500$ *and* $We = 4.6$.

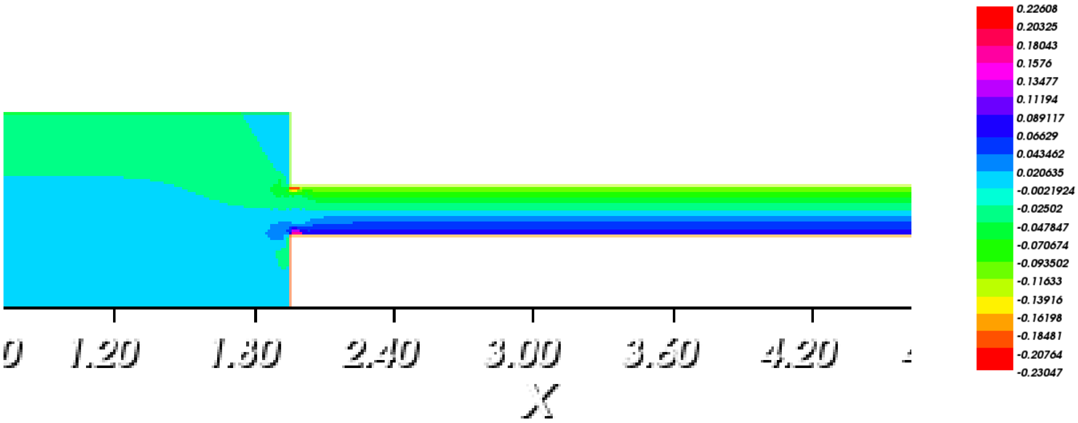


Figure 5.54: *The component of* σ_{12} *(partial view) with* $Re = 500$ *and* $We = 1$.

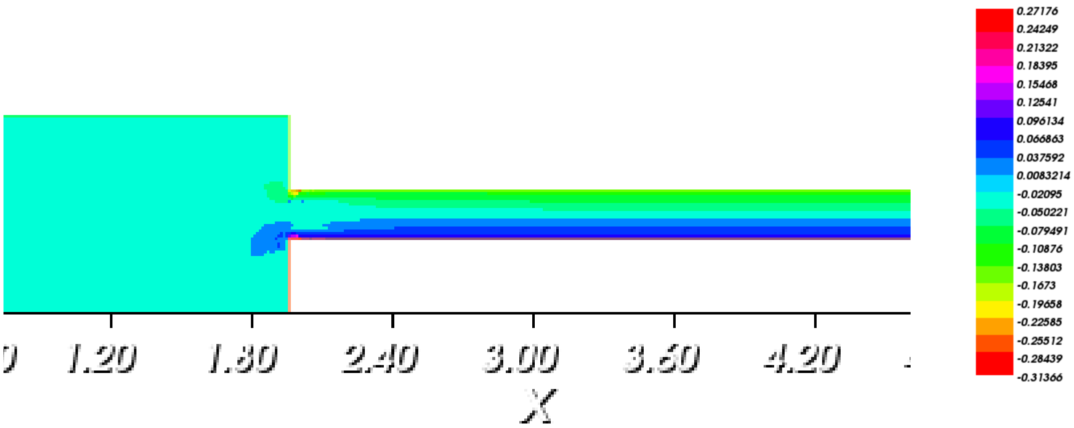


Figure 5.55: *The component of* σ_{12} *(partial view) with* $Re = 500$ *and* $We = 4.6$.

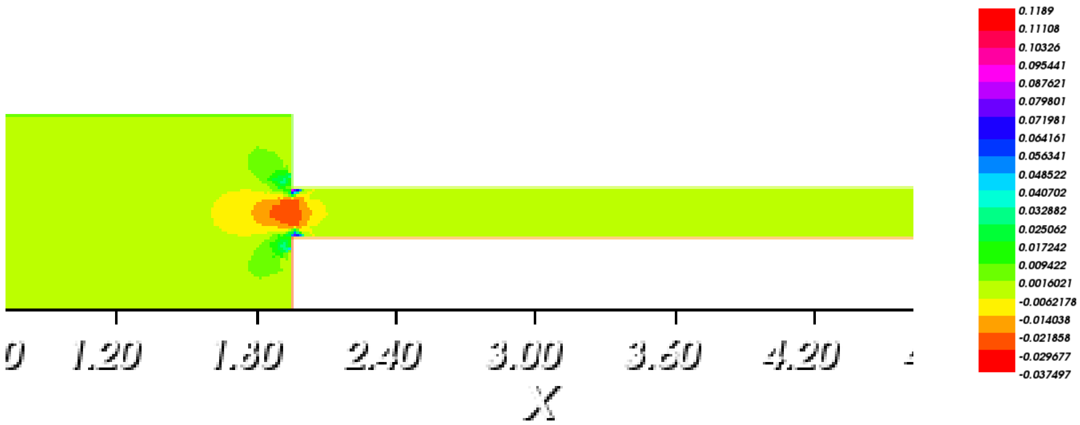


Figure 5.56: *The component of* σ_{22} *(partial view) with* $Re = 500$ *and* $We = 1$.

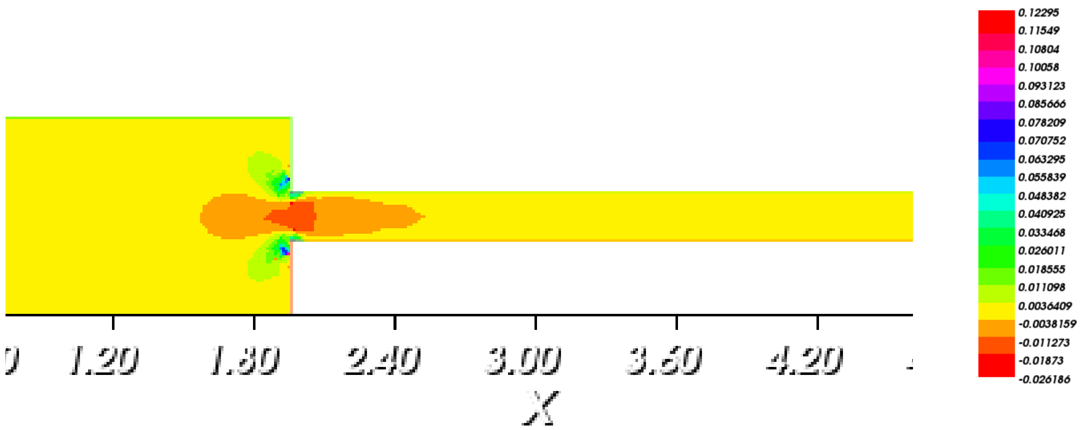


Figure 5.57: *The component of* σ_{22} *(partial view) with* $Re = 500$ *and* $We = 4.6$.

Chapter 6

Conclusions

The main goal of this work was the mathematical and numerical study of the non-linear system of partial differential equations that model the motion of incompressible non-Newtonian fluids of Oldroyd-B type, in dimension 2, in case of steady flow.

We have presented results of existence and uniqueness of the solutions for both problems: the continuous and the discrete problems.

The numerical simulations to the Oldroyd-B problem were obtained computationally by the implementation of the finite elements method (continuous for kinematic and discontinuous for the extra stress tensor) in a script of FreeFem++

This mixed problem of elliptic-hyperbolic type was decoupled into two auxiliary problems, namely, the Navier-Stokes system and the tensorial transport problem. The algorithm to solve the Oldroyd-B problem consists of alternating resolution of transport problem and Navier-Stokes problem.

The Hood-Taylor ($\mathbb{P}_2 - \mathbb{P}_1$) finite elements have been used to discretized the Navier-Stokes system and the iterative Newton-Raphson method has been applied to obtain the numerical solution of the corresponding algebraic system. The discontinuous Galerkin $\mathbb{P}_1$ finite elements were used to solve the transport equation and an iterative fixed-point method type has been applied to obtain the numerical solution of this problem. The validation of numerical methods was made by considering two-dimensional benchmark problems. For the Navier-Stokes, we took a problem with known exact solution and a lid-driven cavity problem.

The approach and discrete problem of Oldroyd-B model were discussed. Based on the numerical techniques described for both the auxiliary problems, the approximation of the solution of the Oldroyd-B problem was obtained and the results of numerical simulations have been presented in two-dimensional case. Numerical results have been obtained in a four-to-one planar contraction (abrupt contraction) for different values of Weissenberg numbers We and Reynolds numbers Re .
We observed viscoelastic behavior of the fluids by comparing the results from the plot of the velocity, pressure and tensor for different values of Re and We. The analysis of the viscosity and the viscoelastic effects was discussed. We have also commented the numerical results for the vortices from the plots of the stream function. For a fixed value of Reynolds number we found a limit for maximum Weissenberg number for which the applied method diverge.

Appendix:

Review on some function spaces

The pair (V, d) consisting of a set V and a metric d is called a metric space. The metric d is a single-valued, nonnegative, real function defined for all $u, v \in V$ which has the following three properties:

- $d(u, v) \geq 0$;
- $d(u, v) = 0$ if and only if $u = v$;
- $d(u, v) = d(v, u)$ (symmetry);
- $d(u, w) \leq d(u, v) + d(v, w) \forall u, v, w \in V$ (triangle inequality).

Let V denotes a real vector space. A real function $\|.\| : V \to \mathbb{R}$ is called a norm if

- $\|u\| = 0$ if and only if $u = 0$ for each $u \in V$;
- $\|\alpha u\| = |\alpha| \|u\|$ for each $u \in V$ and $\alpha \in \mathbb{R}$;
- $\|u + v\| \leq \|u\| + \|v\|$ for every $u, v \in V$.

In fact, norm $\|.\|$ assigns a real number $\|u\|$ to a vector $u \in V$. We have $\|u\| \geq 0$ for each $u \in V$.
A vector space equipped with a norm is called a normed space.

Every normed space $(V, \|.\|_V)$ is a metric space where the metric in V can be defined as $d(u, v) = \|x - y\|_V$.

A sequence $\{u_m\}_{m=1}^{\infty} \subset V$ is said to be *Cauchy sequence* if for every $\epsilon > 0$ there exists a M such that $\|u_m - u_n\| < \epsilon$ for all $m, n > M$, i.e., the sequence satisfies

$$\lim_{n,m \to \infty} ||u_m - u_n||_V = 0$$

Any subset W of a normed space V is called closed if and only if every convergent sequence of elements of W has its limit in W.
The intersection of all closed sets containing $W \subset V$ is called closure of W, and we denote by $cl\ W$.
The subset W of the normed space V is called dense in V if $cl\ W = V$.
A normed space V is called *complete* if each Cauchy sequence in V converges to an element of V.
A complete normed space is said to be a *Banach space*.

The mapping $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is called an inner product or scalar product in V if for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$, the following conditions are satisfied:

- $(u, v) = (v, u)$;
- $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$;
- $(u, u) \geq 0$, and $(u, u) = 0 \Leftrightarrow u = 0$.

A vector space equipped with an inner product is called an inner product space or a pre-Hilbert space.
Every inner product space V is a normed space under the norm defined by

$$\|u\|_V = \sqrt{(u, u)_V}.$$

derived from the inner product. But every normed space may not be an inner product space.

For any two elements u and v of an inner product space V we have the following *Cauchy-Schwarz inequality*:

$$|(u, v)| \leq \|u\|_V \|v\|_V\,, \qquad \forall u, v \in V \tag{A-1}$$

and the following parallelogram law:

$$\|u+v\|_V^2 + \|u-v\|_V^2 = 2\,\|u\|_V^2 + 2\,\|v\|_V^2 \quad \forall u,v \in V. \tag{A-2}$$

An inner product space V is said to be complete if and only if it is a complete metric space under the metric derived from the norm $\|.\|_V = \sqrt{(\cdot,\cdot)_V}$.
A complete inner product space is called a *Hilbert space*. In fact, a Hilbert space is a Banach space endowed with an inner product which generates the norm.

A mapping $l(\cdot) : V \to \mathbb{R}$ is called a linear mapping if

$$l(\alpha v + \beta w) = \alpha l(v) + \beta l(w), \quad \forall\; v,w \in V,\; \forall \alpha,\beta \in \mathbb{R}. \tag{A-3}$$

Let V be a normed spaces. A mapping $l(.) : V \to \mathbb{R}$ is called continuous at $u_0 \in V$ if, for any sequence (u_n) of elements of V convergent to u_0, the sequence $(l(u_n))$ converges to $l(u_0)$, i.e.,

$$\|u_n - u_0\| \to 0 \text{ implies } \|l(u_n) - l(u_0)\| \to 0. \tag{A-4}$$

We simply say that $l(.)$ is continuous, it is continuous at every $u \in V$.
The linear form $l(\cdot)$ is called bounded if there exists a number $\alpha > 0$ such that

$$|l(u)| \leq \alpha ||u||_V, \quad \forall u \in V \tag{A-5}$$

We recall that a linear functional $l(\cdot)$ is continuous if and only if it is bounded.
If the mapping $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ satisfies the following two conditions $\forall \alpha,\beta \in \mathbb{R}$, $\forall u,v,w \in V$:

- $a(\alpha v + \beta w, u) = \alpha a(v,u) + \beta a(w,u)$,
- $a(u, \alpha v + \beta w) = \alpha a(u,v) + \beta a(u,w)$,

then $a(\cdot,\cdot)$ is called bilinear.

Differential Operators, vector and tensor identities and some notations

Kronecker delta

In the representation of mathematical and engineering quantities we frequently use the Kronecker delta which is defined as follows:

$$\delta_{ij} = \delta_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{A-6}$$

The ∇ Operator: Gradient, Divergence, Laplacian and some identities

A vector of the form $(\alpha_1, \cdots, \alpha_n)$ is called a *multi-index* of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where each component α_i is nonnegative integer.
Given a multi-index α, we define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} u.$$

It is usual to think of ∇ as a vector operator defined by

$$\left[\frac{\partial}{\partial x_l}\right]_{l=1,\ldots,3} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}.$$

If $u : \Omega \to \mathbb{R}$ be a scalar function which is differentiable in Ω, then the gradient vector of u is defined by

$$\nabla u = \left[\frac{\partial u}{\partial x_i}\right]_{i=1,\ldots,3} = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{bmatrix}. \tag{A-7}$$

The component of gradient in a direction is the rate of change of u with respect to distance along that direction.

If $\mathbf{u} : \Omega \to \mathbb{R}^3$ is a differentiable vector field then the gradient of $\mathbf{u}$ is defined by

$$\nabla \mathbf{u} = \left[\frac{\partial u_i}{\partial x_j}\right]_{i,j=1,\ldots,3} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}.$$

The gradient of a vector field $\mathbf{u}$ is in fact a tensor field.

Let $\mathbf{u} : \Omega \to \mathbb{R}^3$ be a vector field which is differentiable in Ω. The divergence of $\mathbf{u}$ is the scalar product of ∇ and $\mathbf{u}$ which is defined by

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}.$$

The divergence is a vector operator which produces a scalar value at any point in a vector field.

If $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}$ is a differentiable tensor field, then the divergence of $\boldsymbol{\sigma}$ is a vector field defined by

$$\nabla \cdot \boldsymbol{\sigma} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} = \left[\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}\right]_{i=1,2,3} \tag{A-8}$$

i.e.,

$$(\nabla \cdot \boldsymbol{\sigma})_i = \sum_{j=1}^{n} \partial_j \sigma_{ij}, \quad i = 1, 2, 3 \tag{A-9}$$

The Laplacian operator is a second order differential operator defined by

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \sum_{i=1}^{3} \frac{\partial^2 u_i}{\partial x_i^2}.$$

This is equal to the divergence of the gradient. If $u : \Omega \to \mathbb{R}$ is a twice -differentiable real-valued function then the Laplacian of u is defined by

$$\Delta u = \nabla \cdot (\nabla u) = \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2}. \tag{A-10}$$

The Laplacian can also operate on a vector field. The Laplacian of a vector field $\mathbf{u}$ is defined by

$$\Delta \mathbf{u} = \begin{bmatrix} \Delta u_1 & \Delta u_2 & \Delta u_3 \end{bmatrix}^t = \begin{bmatrix} \sum_{j=1}^{3} \frac{\partial^2 u_1}{\partial x_1^2} \\ \sum_{j=1}^{3} \frac{\partial^2 u_2}{\partial x_2^2} \\ \sum_{j=1}^{3} \frac{\partial^2 u_3}{\partial x_3^2} \end{bmatrix} = \left[\sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2} \right]_{i=1,2,3} \tag{A-11}$$

The derivative of $\mathbf{v}$ in the direction of a unit vector $\mathbf{u}$ is $(\mathbf{u} \cdot \nabla)\mathbf{v}$. Here

$$\mathbf{u} \cdot \nabla = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}$$

For any scalar g and any vector field $\mathbf{u}$,

$$\nabla \cdot (g\mathbf{u}) = g \nabla \cdot \mathbf{u} + \nabla g \cdot \mathbf{u} \tag{A-12}$$

The dyadic product of two tensors is defined as

$$\boldsymbol{\sigma} : \mathbf{T} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} T_{ij} \tag{A-13}$$

So,

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} u_{ij} v_{ij} \tag{A-14}$$

Equalities

If $\mathbf{u}$ be a vector field function on Ω such that $\nabla \cdot \mathbf{u} = 0$, then

$$2\nabla \cdot \mathbf{D}(\mathbf{u}) = 2\nabla \cdot \left[\frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t \right) \right] = \nabla \cdot (\nabla \mathbf{u}) + \nabla \cdot (\nabla \mathbf{u})^t = \Delta \mathbf{u} + \nabla \cdot (\nabla \mathbf{u})^t. \tag{A-15}$$

But we have

$$\left[\nabla\cdot(\nabla\mathbf{u})^t\right]_i=\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i}=\sum_{j=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}=\frac{\partial}{\partial x_i}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j}=\left[\nabla\left(\nabla\cdot\mathbf{u}\right)^t\right]_i \qquad \text{(A-16)}$$

So, $\nabla\cdot(\nabla\mathbf{u})^t=\nabla\left(\nabla\cdot\mathbf{u}\right)$.

Since $\nabla\cdot\mathbf{u}=0$, from $(A-15)$, we have

$$2\nabla\cdot\mathbf{D}\left(\mathbf{u}\right)=2\nabla\cdot\left[\frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^t\right)\right]=\Delta\mathbf{u}+\nabla\left(\nabla\cdot\mathbf{u}\right)=\Delta\mathbf{u}. \qquad \text{(A-17)}$$

Bibliography

[1] R. A. Adams and John J. F. Fournier, *Sobolev Spaces*,Second Edition, Academic Press, New York (2003).

[2] N. Arada and A. Sequeira, *Strong steady solutions for a generalized Oldroyd-B model with shear-dependent viscosity in a bounded domain*, Math. Models Methods Appl. Sci., 13, no. 9, 1303-1323, (2003).

[3] E. Becker, G. Carey and J. Oden, *Finite Elements. An Introduction*, Vol.1, Prentice Hall, Inc., Englewood Cliffs, New Jersey (1981).

[4] R. B. Bird, R. C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids*, Vol.1, Fluid Mechanics, John Wiley & Sons, Second Edition, (1987).

[5] L. Borges, *Domain Decomposition Methods for Non-Newtonian Fluids*, PhD Thesis, IST, Lisbon (2006).

[6] S.Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York (1994)

[7] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2011).

[8] F. Brezzi, L. D. Marini and E. Süli, *Discontinuous Galerkin methods for first-order hyperbolic problems, Sobolev Spaces and Partial Differential Equations*, Springer (2011).

[9] G. F. Carey and J. T. Oden, *Finite elements. Vol. VI. Fluid mechanics.*, The Texas Finite Element Series, VI. Prentice Hall, Inc., Englewood Cliffs, New Jersey (1986).

[10] P. G. Ciarlet, *Mathematical Elasticity: Three-dimensional elasticity, Vol. 1* Springer-Verlag, North-Holland (1988).

[11] A. J. Chorin and J. E. Marsden , *A Mathematical Introduction to Fluid Mechanics*, 3ed., Springer, (2000)(179s).

[12] B. Cockburn, G. Karniadakis and C. Shu, *Discontinuous Galerkin Methods-Theory, Computation and Applications*, Vol. 11 of Lecture Notes in Computer Science and Engineering, Springer, 2000.

[13] P. Correia, *Numerical Simulations of a Non -Newtonian Fluid Flow Model Using Finite Element Methods*,PhD Thesis, IST, Lisbon (2004).

[14] L. Debnath and P. Mikusiński, *Hilbert Spaces with Applications*, Third Edition, Elsevier Academic Press,California (2005).

[15] A. Ern and J. Guermond, *Discontinuous Galerkin Methods for Friedrichs' Systems. I. General Theory*, SIAM J. Numer. Anal, 44(2), 753-778 (2006).

[16] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearized Steady Problems,* Springer Tracts in Natural Philosophy, Vol. 38, Springer, New York (1994).

[17] U. Giha, K.N. Ghia and and C. T. Shin, *High-resulations incompressible flows using Navier-Stokes equations and a multigrid method*, J. Comput. Phys. 1982, 48.

[18] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Math., 749, Springer-Verlag, Berlin (1979).

[19] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations-Theory and Algorithms*, Springer-Verlag (1986).

[20] C. Guillopé and J. C. Saut, *Existence and stability of steady flows of weakly viscoelastic fluids*, Proc. Royal Soc. Edinburgh, 119 A, pp. 137-158, (1991).

[21] A. S. Gupta and K. R. Rajagopal, *An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate*, Mechanica, 19, 158-160, (1984).

[22] J. Hecht, *FreeFem++*, Version 3.23, 2012, http://www.freefem.org/ff++.

[23] J. Hron, *Numerical Simulation of Visco-Elastic Fluids*, In: WDS' 97, Freiburg, (1997).

[24] C. Johnson, *Numerical Solutions of Partial Differential Equations by the Finite Element Method*, Cambridge University Press (1987).

[25] P. N. Kaloni and K. R. Rajagopal, *Some remarks on boundary conditions for flow of fluids of the differential type*, in Continuum Mechanics and its Applications, Hemisphere Press, New York, NY (1989).

[26] S. D. Kim, Y. H. Lee and B. C. Shin, *Newton's Method for the Navier-Stokes Equations with Finite-Element Initial Guess of Stokes Equation*, Computers and Mathematics with applications, 51, 805-816 (2006).

[27] P. Lesaint and P. A. Raviart, *On a finite element method for solving the neutron transport equation, C. Boor (editor), Mathematical Aspects of Finite Elements in Partial Differential Equations*, 89-123, New York, Academic press, (1974).

[28] B. Q. Li, *Discontinuous Finite Elements in Fluid Dynamics and Heat Transfer*, Springer-Verlag (2006).

[29] B. Lucquin and O. Pironneau, *Introduction au Calcul Scientifique,* Masson (1996).

[30] K. Najib and D. Sandri, *On a decoupled algorithm for solving a finite element problem for the approximation of viscoelastic fluid flow*, Numer. Math. 72, 223-238, (1993).

[31] Y. Nakayama and R. F. Boucher, *Introduction to Fluid Mechanics*, Butterworth-Heinemann (2000).

[32] A. Novotný, *About the steady transport equation I - L^p-Approach in domains with smooth boundaries*, Comm. Mat. Univ. Carolinae, 37, 43-89, 1996.

[33] M. Pires, *Mathematical and Numerical Analysis of Non-Newtonian Fluids in Curved Pipes*, PhD Thesis, IST, Lisbon (2005).

[34] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer-Verlag (1994).

[35] A. Quarteroni and L. Formaggia, *Mathematical Modelling and Numerical Simulation of the Cardiovascular System,* Elsevier, Amsterdam(2002).

[36] K. R. Rajagopal, *On boundary conditions for fluids of differential type, A. Sequeira (ed.) Navier-Stokes Equations and Related Non-Linear Problems*, Plenum Press, 273-278, (1995).

[37] P. A. Raviart and J. M. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson (1983).

[38] M. Renardy, *Mathematical Analysis of Viscoelastic Flows*, SIAM (2000).

[39] M. Renardy, *Existence of Slow Steady Flows of Viscoelastic Fluids with Differential Constitutive Equations*, Z. Angew. Math. u Mech., 65, pp. 449-451, (1985).

[40] M. Renardy, *Recent advances in the mathematical theory of steady flow of viscoelastic fluids*, J. of Non-Newtonian Fluid Mechanics, 9, pp. 11-24, (1988).

[41] Y. Saad, *Iterative Methods for Sparse Linear Systems*, Second edition, Society for Industrial and Applied Mathematics, Philadelphia (2003).

[42] P. Saramito, *A new θ-scheme algorithm and incompressible FEM for viscoelastic fluid flows*, Mathematical Modelling and Numerical Analysis, vol. 28, n. 1, pp. 1-34, 1994.

[43] P. Saramito, *Simulation numérique découlements de fluides visco-élastiques par éléments finis incompressibles et une méthode de directions alternés Applications*, Thèse de l'Institut National Polytechnique de Grenoble (1990).

[44] R. Schreiber, H. B. Keller, *Driven cavity flows by efficient numerical techniques*, J. Comput. Phys. 1983, 49.

[45] J. Serrin, *Mathematical Principles of Classical Fluid Mechanics*, in S. Flugge and S. Truesdell, editors, Handbuch der Physik, Vol. VIII, Springer-Verlag, Berlin (1969).

[46] E. J. Shaughnessy, I. M. Katz and J. P. Schaffer, *Introduction to Fluid Mechanics,* Oxford University Press (2005)

[47] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, 3rd ed.North-Holland, Amsterdam (1984).

[48] H. B. Veiga, *On a stationary transport equation*, Ann. University Ferrara, Sez. VII- Sc.Mat, XXXII 79-91, (1986).

[49] H. B. Veiga, *Existence results in Sobolev spaces for a stationary transport equation*, Ricerch a di Matematica, 36, 173-184, (1987).

[50] J. H. Videman, *Mathematical Analysis of Viscoelastic non-Newtonian Fluids*, PhD Thesis, IST, Lisbon (1997).

[51] J. Yeh, *Real Analysis, Theory of Measure and Integration*, Second Edition, World Scientific Publishing Co. Pte. Ltd. (2000).

[52] K. Yosida, *Functional Analysis*, Sixth Edition, Springer-Verlag (1980).