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Constructive Decomposition of Any $L^1(a,b)$ Function as Sum of a Strongly Convergent Series of Integrable Functions Each One Positive or Negative Exactly in Open Sets

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Abstract. Researchers dealing with real functions $f(\cdot) \in L^1(a,b)$ are often challenged with technical difficulties on trying to prove statements involving the positive $f^+(\cdot)$ and negative $f^-(\cdot)$ parts of these functions. Indeed, the set of points where $f(\cdot)$ is positive (resp. negative) is just Lebesgue measurable, and in general these two sets may both have positive measure inside each nonempty open subinterval of (a,b). To remedy this situation, we regularize these sets through open sets. More precisely, for each zero-average $f(\cdot) \in L^1(a,b)$, we construct, explicitly, a series of functions $\widehat{f}_i(\cdot)$ having sum $f(\cdot)$ — a.e. and in $L^1(a,b)$ — in such a way that, for each $i \in \{0,1,2,\dots\}$, there exist two disjoint open sets where $\widehat{f}_i(\cdot) \geq 0$ a.e. and $\widehat{f}_i(\cdot) \leq 0$ a.e., respectively, while $\widehat{f}_i(\cdot) = 0$ a.e. elsewhere. Moreover, its primitive $\int^t f(\cdot)$ becomes the sum of a strongly convergent series of nice AC functions. Applications to calculus of variations & optimal control appear in our next papers.

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1. Introduction

The main focus of this paper is on functions

$$f(\cdot) \in L^1(a,b) \quad \text{having} \quad \int_a^t f(\tau)\,\mathrm{d}\tau \geq 0 \quad \forall t \quad \& \quad \int_a^b f(t)\,\mathrm{d}t = 0, \quad (1.1)$$

i.e. each $f:(a,b)\subset\mathbb{R}\to[-\infty,\infty]$ (with $a<b$) is a Lebesgue-integrable function with positive primitive and zero average. Or, equivalently, their primitives

$$\begin{aligned}
&F(t):=\int_a^t f(\tau)\,\mathrm{d}\tau\geq 0 \ \ \forall\, t, \quad \text{hence}\\
&F'(t)=\frac{\mathrm{d}}{\mathrm{d}t}F(t)=f(t) \ \text{ a.e. on } [a,b]\,, \quad \text{satisfy:}\\
&F(\cdot)\in W_0^{1,1}(\,[a,b]\,)\\
&(\text{i.e. } F(\cdot) \text{ is AC (absolutely continuous) with } F(a)=0=F(b)).
\end{aligned}$$

After dealing with such positive functions, we also treat negative functions, hence vectorial functions in general, in (1.9)–(1.12) and from (2.35) on.

But consider first the special case of (1.1) in which

$$\exists\, c\in(a,b):\ f(t)\geq 0 \ \text{ a.e. in } (a,c) \quad \& \quad f(t)\leq 0 \ \text{ a.e. in } (c,b)\,,$$

so that

$$\begin{aligned}
&F(\cdot)\geq 0 \quad \text{is a } \textit{cap function, i.e. } F(a)=0=F(b) \ \text{ and}\\
&F(\cdot)\,|_{[a,c]} \ \text{ increases while } \ F(\cdot)\,|_{[c,b]} \ \text{ decreases.}
\end{aligned} \tag{1.2}$$

If we disregard what has zero measure, in this simpler situation the positivity and negativity sets

$$E_f^+:=\{\,t\in(a,b):\ f(t)>0\,\} \quad \& \quad E_f^-:=\{\,t\in(a,b):\ f(t)<0\,\}$$

certainly are contained in disjoint open sets:

$$E_f^+\subset(a,c) \quad \& \quad E_f^-\subset(c,b) \quad \text{a.e.}.$$

Such neat separation between these two sets E_f^+ & E_f^- gets of course lost in the general case (1.1) of wilder zero-average integrable functions having positive indefinite integral, to the point that E_f^+ & E_f^- may even both have positive measure over each nonempty open subinterval of (a,b) — i.e. will be inextricably intertwined. (A possibility which easily follows from the Borel set mentioned in [8, Chapter II, problem 8].)

Aim of this paper is precisely to fix this issue, namely by explicitly constructing a decomposition for each $f(\cdot)$ satisfying (1.1), so that it becomes the sum of a L^1-strongly and a.e. convergent series of functions $\widehat{f}_i(\cdot)$, each one again satisfying (1.1) together with:

$$\begin{aligned}
&\exists \text{ disjoint open sets } A_i^+, A_i^- \text{ for which:}\\
&\widehat{f}_i(\cdot)\geq 0 \ \text{ a.e. in } A_i^+ \quad \& \quad \widehat{f}_i(\cdot)\leq 0 \ \text{ a.e. in } A_i^-\\
&\textstyle\int_a^t \widehat{f}_i(\tau)\,\mathrm{d}\tau=0 \ \text{ at a.e. } t\in Z_i:=[a,b]\setminus\left(A_i^+\cup A_i^-\right);\\
&\text{while the measure}
\end{aligned} \tag{1.3}$$

$$\left|\left\{\,t\in[a,b]:\ \sum_{i=0}^{k}\widehat{f}_i(t)\neq f(t)\right\}\right| \quad \xrightarrow[k\to\infty]{} \quad 0. \tag{1.4}$$

Detailing the convergence of this series, it is such that, besides

$$f(t) = \sum_{i=0}^{\infty} \widehat{f}_i(t) \quad \text{for a.e. } t \in [a,b], \tag{1.5}$$

we also have

$$\int_a^b \left| f(t) - \sum_{i=0}^{k} \widehat{f}_i(t) \right| \mathrm{d}t \quad \xrightarrow[k \to \infty]{} \quad 0. \tag{1.6}$$

As we shall see, this result may also be interpreted as follows: given any AC function

$$F(\cdot) \in W_0^{1,1}([a,b]) \quad \text{having} \quad F(\cdot) \geq 0 \tag{1.7}$$

and defining

$$f(t) := F'(t) \text{ a.e.},$$

we explicitly construct a decomposition of $F(\cdot)$ as the sum of a strongly-$W^{1,1}$ convergent series of functions $\widehat{F}_i(\cdot)$ $\left(:= \int \widehat{f}_i(\cdot)\right)$, each one also satisfying (1.7), such that

> $\widehat{F}_i(\cdot)$ restricted to each maximal nonempty interval $\left(a_i^j, b_i^j\right)$ of the open set $\left\{ t \in (a,b) : \widehat{F}_i(t) > 0 \right\}$ is a cap function (i.e. satisfies (1.2) for an adequate c_i^j);

while the measure

$$\left| \left\{ t \in [a,b] \setminus C_F : \sum_{i=0}^{k} \widehat{F}_i(t) \neq F(t) \right\} \right| \quad \xrightarrow[k \to \infty]{} \quad 0. \tag{1.8}$$

Here $C_F := \{ t \in [a,b] : \exists F'(t) = 0 \}$ is the critical set of the given function $F(\cdot)$ (which often has nonempty interior, due to constancy intervals of $F(\cdot)$); and the reason why one has to exclude C_F from (1.8), but not from (1.4), is that while our construction always yields

$$\sum_{i=0}^{k} \widehat{F}_i^{\,\prime}(t) = 0 = F^{\prime}(t) \quad \text{a.e. on } \ C_F \quad \forall\, k \in \{0,1,2,\dots\},$$

$$\text{it may well happen} \quad \sum_{i=0}^{k} \widehat{F}_i(t) < F(t) \quad \text{a.e. on } \ C_F \quad \forall\, k \in \{0,1,2,\dots\},$$

e.g. when C_F has nonempty interior and each function $\widehat{F}_i(\cdot)$ has constant > 0 value along each interval of C_F.

In case Fourier expansions come to the reader's mind as a possibility to get (1.5), let's recall that Fourier series for L^1 functions may even diverge at all points. (Kolmogorov constructed such an example in [5].)

On the other hand, Fourier expansions will in general not satisfy the very useful convergence property (1.8). In fact, starting e.g. from a piecewise-affine cap-function $F(\cdot)$ one reaches a trigonometrical Fourier series with infinitely many nonzero terms; while our cap-series will, by construction, contain just one nonzero term $\widehat{F}_0(\cdot) = F(\cdot)$, since $F(\cdot)$ is itself already a cap-function.

Thus the usual approach of harmonic analysis — consisting in fixing e.g. an orthonormal basis in a functional space hence express its functions relative to such basis — is completely different from our own. Indeed, defining $F_0(\cdot) := F(\cdot)$, our strategy begins instead with the explicit construction, along each maximal interval where $F_0(\cdot) > 0$, of a very reasonable cap function $\widehat{F}_0(\cdot)$ which equals $F_0(\cdot)$ in (at least) part of that interval; and then we do the same to $F_{i+1}(\cdot) := \left(F_i - \widehat{F}_i\right)(\cdot)$, for $i = 0, 1, \dots$. In particular, our series will contain infinitely many nonzero terms only in case $F(\cdot)$ does consist of infinitely many levels of superposed oscillations (over the bounded interval (a,b)). In the special case where $F(\cdot)$ is not only AC but even analytic, our series will thus become just a finite sum of finitely-piecewise-cap functions, namely $\exists\, k_F \in \mathbb{N}$ such that

$$F(t) = \widehat{F}_0(t) + \widehat{F}_1(t) + \dots + \widehat{F}_{k_F}(t) \quad \forall\, t \in [a,b], \text{ with each open set}$$
$$\left\{ t \in (a,b) : \widehat{F}_i(t) > 0 \right\} \text{ having finitely many connected components.}$$

Such decomposition of AC functions into cap components is here presented as a technical tool allowing research mathematicians to deal, in an easier way, with positive or negative parts of L^1 functions and with monotonicity intervals of AC functions. Ourselves — in past research on nonconvex calculus of variations & optimal control (see e.g. [1,2,7]) — have felt technical difficulties on dealing with sets of positivity and of negativity of scalar velocities, since these may well have positive measure boundaries. But our new decomposition (1.3) and (1.4) allows us to reach better nonconvex results, and even results on the sign of integrals, in new papers to appear.

On the other hand, using as building blocks our cap-decomposition for scalar positive $W_0^{1,1}([a,b])$ functions, one easily reaches more general formulations. Indeed, the vectorial versions of our above decompositions for L^1 and for AC scalar functions, respectively, turn out to be the following:

Corollary 1. *Given any function* $f(\cdot) \in L^p((a,b),\mathbb{R}^m)$ $(1 \le p < \infty)$ *let*

$$m_f := \frac{1}{b-a}\int_a^b f(t)\,dt, \quad \text{so that } f(\cdot) - m_f \text{ has zero average.} \tag{1.9}$$

Then we explicitly construct, from $f(\cdot)$, *zero-average functions* $\widehat{f}_i(\cdot) \in L^p((a,b),\mathbb{R}^m)$ *such that*

$$\sum_{i=0}^{\infty} \widehat{f}_i(\cdot) = f(\cdot) - m_f \text{ a.e and in } L^p((a,b),\mathbb{R}^m),$$

$$\left(\sum_{i=0}^{k} \widehat{f}_i(\cdot) - [f(\cdot) - m_f]\right) \xrightarrow[k\to\infty]{} 0 \quad \text{in } L^\infty((a,b)\backslash E,\mathbb{R}^m)$$

with measure $|E|$ *arbitrarily small,*

$$\left|\sum_{i=0}^{k} \widehat{f}_i(\cdot) - [f(\cdot) - m_f]\right|_{L^\infty((a,b)\backslash E_k,\mathbb{R}^m)} = 0$$

with measure $|E_k| \xrightarrow[k\to\infty]{} 0$, *since the measure*

$$\left|\left\{t \in [a,b]: \sum_{i=0}^{k} \widehat{f}_i(t) \neq f(t) - m_f\right\}\right| \xrightarrow[k\to\infty]{} 0. \tag{1.10}$$

Moreover, each coordinate of each $\widehat{f}_i(\cdot)$ *satisfies* (1.3).

Corollary 2. *Given any function* $F(\cdot) \in W^{1,p}([a,b],\mathbb{R}^m)$ $(1 \le p < \infty)$ *let*

$$M_F(t) := F(a) + \frac{t-a}{b-a}[F(b) - F(a)], \quad \text{so that}$$
$$F(\cdot) - M_F(\cdot) \in W_0^{1,p}([a,b],\mathbb{R}^m);$$

and consider the critical set:

$$C_F := \{t \in [a,b]: \ \Pi(F'(t) - M_F'(t)) = 0\}, \text{ where}$$
$$\Pi : \mathbb{R}^m \to \mathbb{R} \text{ is the product } \Pi(s) = \Pi(s^1, s^2, \ldots, s^m) := s^1 . s^2 . \ldots . s^m.$$

Then we explicitly construct, from $F(\cdot)$, *functions* $\widehat{F}_i(\cdot) \in W_0^{1,p}([a,b],\mathbb{R}^m)$ *such that*

$$\sum_{i=0}^{\infty} \widehat{F}_i(\cdot) = F(\cdot) - M_F(\cdot) \quad \text{in } C^0([a,b],\mathbb{R}^m) \text{ and in } W_0^{1,p}([a,b],\mathbb{R}^m),$$

$$\left(\sum_{i=0}^{k} \widehat{F}_i(\cdot) - [F(\cdot) - M_F(\cdot)]\right) \xrightarrow[k\to\infty]{} 0 \quad \text{in } W^{1,\infty}([a,b]\setminus E,\mathbb{R}^m)$$

with measure $|E|$ *arbitrarily small,*

$$\left|\sum_{i=0}^{k} \widehat{F}_i(\cdot) - [F(\cdot) - M_F(\cdot)]\right|_{W^{1,\infty}(([a,b]\setminus C_F)\setminus E_k,\mathbb{R}^m)} = 0$$

with measure $|E_k| \xrightarrow[k\to\infty]{} 0$, *due to* (1.10) *and the measure*

$$\left|\left\{ t \in [a,b] \setminus C_F : \sum_{i=0}^{k} \overset{\frown}{F}_i(t) \neq F(t) - M_F(t) \right\}\right| \underset{k \to \infty}{\longrightarrow} 0. \quad (1.11)$$

Moreover, each coordinate of each $\overset{\frown}{F}_i(\cdot)$ *is a countably-piecewise-cap/cup function.*

(What we mean here is that each coordinate of each $\overset{\frown}{F}_i(\cdot)$ is — locally where it is $\neq 0$ — either a cap function or a cup function, calling

$$\begin{array}{l} F(\cdot) \leq 0 \text{ a } \textit{cup function} \text{ whenever} \\ -F(\cdot) \text{ is a cap function in the sense (1.2).)} \end{array} \quad (1.12)$$

2. Cap or Cup Decomposition of AC Functions

To begin with, associating to each function

$$\begin{array}{l} F(\cdot) \in W_0^{1,p}([a,b]) \quad (1 \leq p < \infty) \quad \text{with} \\ F(\cdot) \geq 0 \quad \& \quad F(\cdot) \not\equiv 0 \end{array} \quad (2.1)$$

the corresponding open set

$$\emptyset \neq \mathcal{O}_0 := \{ t \in (a,b) : F(t) > 0 \} = \bigcup_{j \in J_0} \left(a_0^j, b_0^j \right), \quad J_0 \subset \mathbb{N} \quad (2.2)$$

$\Big($with disjoint nonempty intervals $\left(a_0^j, b_0^j \right)\Big)$, then here is our main result:

Theorem 1. (Cap decomposition of positive periodic AC functions) *For each function* $F(\cdot)$ *as in* (2.1) *and* (2.2) *there exist functions* $\overset{\frown}{F}_i(\cdot)$, *each one a countably-piecewise-cap function, such that, for each* $q \in [1,p]$,

$$F(\cdot) = \sum_{i=0}^{\infty} \overset{\frown}{F}_i(\cdot) \quad \textit{in} \ W_0^{1,q}([a,b]), \quad \textit{hence uniformly,} \quad (2.3)$$

together with

$$\begin{array}{l} f(\cdot) := F'(\cdot) = \sum_{i=0}^{\infty} \overset{\frown}{f}_i(\cdot) \quad \textit{in} \ L^q(a,b) \ \textit{and a.e.,} \\ \textit{hence almost uniformly, with} \ \overset{\frown}{f}_i(\cdot) := \overset{\frown}{F}_i{}'(\cdot) \ \textit{a.e..} \end{array} \quad (2.4)$$

More precisely: (1.5) and (1.6) hold true, with $|\cdot|$ replaced by $|\cdot|^q$, and — using the maximal intervals $\left(a_i^j, b_i^j\right)$ of the open set $\mathcal{O}_i$ where $\widehat{F}_i(\cdot) > 0$ $(i \in I \subset \mathbb{N} \cup \{0\})$ — each $\widehat{F}_i(\cdot)$ will be " *a cap function along each* $\left(a_i^j, b_i^j\right)$ ", i.e.

$$\exists\, c_i^j \in \left(a_i^j, b_i^j\right) \quad (j \in J_i \subset \mathbb{N}):$$

$$\widehat{f}_i(\cdot) \geq 0 \text{ a.e. in } \left(a_i^j, c_i^j\right) \quad \& \quad \widehat{f}_i(\cdot) \leq 0 \text{ a.e. in } \left(c_i^j, b_i^j\right).$$

The idea behind the proof of Th. 1 is the following: after extracting from $F_0(\cdot) := F(\cdot)$ a first cap level $\widehat{F}_0(\cdot)$ (= its cap components), we then extract from $F_1(\cdot) := F_0(\cdot) - \widehat{F}_0(\cdot)$ a second level $\widehat{F}_1(\cdot)$ (= its cap subcomponents), and so on and so forth ad infinitum. Thus $\widehat{F}_0(\cdot)$ (resp. $\widehat{F}_1(\cdot)$, $\widehat{F}_2(\cdot)$, ...) yields the cap components (resp. subcomponents, subsubcomponents, ...) of $F_0(\cdot)$; meaning, by "subcomponents" (resp. by "subsubcomponents", ...), that $\widehat{F}_1(\cdot)$ (resp. $\widehat{F}_2(\cdot)$, ...) yields the cap components of $F_1(\cdot)$ (resp. of $F_2(\cdot) := F_1(\cdot) - \widehat{F}_1(\cdot)$, ...). While this strategy is simple to explain, the real challenge has been how to prove, in a constructive way, strong L^1-convergence, to zero, of derivatives (as in (2.24)). Notice that since our cap decomposition is explicitly constructed, step by step, it wouldn't be so nice (both aesthetically and from the pragmatic viewpoint of practical implementation of the decomposition) to end up such construction with a nonconstructive proof of convergence.

Proof of Theorem 1. The promised decomposition (2.3) — or (2.4) or, in particular, (1.6) — will be constructed as follows: taking any

$$F(\cdot) \in W_0^{1,p}([a,b]) \quad \text{having} \quad F(\cdot) \geq 0 \quad \& \quad \text{satisfying (2.2)},$$

one defines, recursively,

$$\begin{array}{l} F_{i+1}(\cdot) := F_i(\cdot) - \widehat{F}_i(\cdot), \quad \text{for } i = 0, 1, 2, \dots, \quad \text{with} \\ F_0(\cdot) := F(\cdot), \end{array} \tag{2.5}$$

where $\widehat{F}_i(\cdot)$, the union of the disjoint *cap components* of $F_i(\cdot)$, is defined as follows: in case

$$\mathcal{O}_i := \{\, t \in (a,b) : \; F_i(t) > 0 \,\} \tag{2.6}$$

is nonempty, we set

$$\widehat{F}_i(t) := \begin{cases} \min F_i\left(\left[t, c_i^j\right]\right) & \text{for } t \in \left[a_i^j, c_i^j\right] \\ \min F_i\left(\left[c_i^j, t\right]\right) & \text{for } t \in \left[c_i^j, b_i^j\right] \\ 0 & \text{for } t \in [a,b] \setminus \mathcal{O}_i, \end{cases} \tag{2.7}$$

with

$$\left(a_i^j, b_i^j\right),\ j \in J_i\,, \quad \text{the maximal nonempty intervals of } \mathcal{O}_i,$$
$$\text{while} \quad c_i^j := \min\left\{ t \in \left[a_i^j, b_i^j\right] : F_i(t) = \max\, F_i\left(\left[a_i^j, b_i^j\right]\right)\right\};$$

otherwise, i.e. if $\mathcal{O}_i = \emptyset$, we put

$$\widehat{F}_i(\cdot) := F_i(\cdot)\,. \tag{2.8}$$

Considering the set of relevant i's,

$$I := \{\, i \in \{0,\, 1,\, 2,\, \ldots\} :\ \mathcal{O}_i \neq \emptyset \,\},$$

then clearly, for $i \in I$,

$$\begin{aligned} &\widehat{F}_i\left(a_i^j\right) = F_i\left(a_i^j\right) = F_i\left(b_i^j\right) = \widehat{F}_i\left(b_i^j\right) = 0 \\ &\leq \widehat{F}_i(t) \leq F_i(t) \leq \widehat{F}_i\left(c_i^j\right) = F_i\left(c_i^j\right) \\ &\forall\, t \in \left[a_i^j, b_i^j\right] \quad \forall\, j \in J_i \end{aligned} \tag{2.9}$$

so that, by (2.5), $0 \leq F_i(t) - \widehat{F}_i(t) = F_{i+1}(t) \leq F_i(t)$ and

$$\begin{aligned} &0 \leq F_{i+1}(t) \leq F_i(t) \leq \max\, \widehat{F}_i([a,b]) = \\ &= \max\, F_i([a,b]) \quad \forall\, t \in [a,b]\,. \end{aligned} \tag{2.10}$$

Recalling now the open set $\mathcal{O}_i$ in (2.6), define the new functions and sets of regular points (see (2.4))

$$f_i(\cdot) := F_i'(\cdot)\,, \quad R_i := \{\, t \in \mathcal{O}_i :\ \exists\, f_i(t) \neq 0 \,\} \quad \& \\ \widehat{R}_i := \left\{ t \in \mathcal{O}_i :\ \exists\, \widehat{f}_i(t) \neq 0 \right\}$$

obtaining, by (2.7), using the characteristic functions of $\widehat{R}_i$ & R_i,

$$\widehat{f}_i(t) = f_i(t)\, \chi_{\widehat{R}_i}(t) \ \text{ a.e.} \quad \& \quad \chi_{\widehat{R}_i}(t) \leq \chi_{R_i}(t) \ \text{ a.e..} \tag{2.11}$$

On the other hand, differentiating (2.5),

$$\begin{aligned} f_{i+1}(t) &= f_i(t) - \widehat{f}_i(t) = f_i(t)\, \chi_{R_i}(t) - f_i(t)\, \chi_{\widehat{R}_i}(t) = \\ &= f_i(t)\, \chi_{R_i \setminus \widehat{R}_i}(t) \ \text{ a.e.,} \end{aligned}$$

in particular $|\, f_{i+1}(t)\,|^q = |\, f_i(t)\,|^q\, \chi_{R_i \setminus \widehat{R}_i}(t) = |\, f_i(t)\,|^q \left(1 - \chi_{\widehat{R}_i}\right)(t) = |\, f_i(t)\,|^q - \left|\, \widehat{f}_i(t) \right|^q$ a.e. and

$$f_i(t) = \widehat{f}_i(t) + f_{i+1}(t) \quad \& \quad |\, f_i(t)\,|^q = \left|\, \widehat{f}_i(t) \right|^q + |\, f_{i+1}(t)\,|^q \quad \text{a.e.} \\ \chi_{R_{i+1}}(t) = \chi_{R_i \setminus \widehat{R}_i}(t) \leq \chi_{R_i}(t) \ \text{ a.e..} \tag{2.12}$$

Applying recursively these equalities one obtains, by (2.11),

$$f_i(t) = f(t)\, \chi_{R_i}(t) \ \text{ a.e.} \quad \& \quad \widehat{f}_i(t) = f(t)\, \chi_{\widehat{R}_i}(t) \quad \text{a.e.} \tag{2.13}$$

$$\begin{aligned} f(t) &= f_{N+1}(t) + \sum_{i=0}^{N} \widehat{f}_i(t) \ \text{ a.e.} \quad \& \\ |\, f(t)\,|^q &= |\, f_{N+1}(t)\,|^q + \sum_{i=0}^{N} \left| \widehat{f}_i(t) \right|^q \ \text{ a.e.} \end{aligned} \tag{2.14}$$

hence, integrating along $[a,b]$ and letting $N \to \infty$, using (2.14) and (2.13) one gets

$$\begin{aligned} \int_a^b |\, f(t)\,|^q \,\mathrm{d}t &= \int_a^b \sum_{i=0}^{\infty} \left| \widehat{f}_i(t) \right|^q \mathrm{d}t + \lim_{i\to\infty} \int_a^b |\, f_i(t)\,|^q \,\mathrm{d}t \\ &= \sum_{i=0}^{\infty} \int_a^b \left| \widehat{f}_i(t) \right|^q \mathrm{d}t + \lim_{i\to\infty} \int_a^b |\, f(t)\,|^q \,\chi_{R_i}(t)\,\mathrm{d}t, \end{aligned}$$

by monotone convergence; which means

$$|\, f(\cdot)\,|^q_{L^q(a,b)} = \sum_{i=0}^{\infty} \left| \widehat{f}_i(\cdot) \right|^q_{L^q(a,b)} + \lim_{i\to\infty} |\, f_i(\cdot)\,|^q_{L^q(a,b)}, \tag{2.15}$$

in particular

$$\begin{aligned} &\left(|\, f_i(\cdot)\,|_{L^q(a,b)} \right) \text{ converges} \quad \& \\ &\left(\left| \widehat{f}_i(\cdot) \right|_{L^q(a,b)} \right) \to 0, \ \text{ as } \ i \to \infty. \end{aligned} \tag{2.16}$$

More precisely than in the *lhs* of (2.16) we have, by (2.13) and (2.12),

$$(\, f_i(t)\,) \to m(t) := f(t)\,\chi_R(t) \ \text{ for a.e. } t, \ \text{ where } \ R := \bigcap_{i=0}^{\infty} R_i; \tag{2.17}$$

hence Lebesgue dominated convergence yields, since $|\, f_i(\cdot)\,|^q \le |\, f(\cdot)\,|^q$ a.e. $\forall\, i$,

$$\lim_{i\to\infty} F_i(t) = \lim_{i\to\infty} \int_a^t f_i(\tau)\,\mathrm{d}\tau = \int_a^t m(\tau)\,\mathrm{d}\tau \quad \forall\, t \in [a,b] \tag{2.18}$$

$$\text{and} \quad \lim_{i\to\infty} |\, f_i(\cdot)\,|^q_{L^q(a,b)} = \lim_{i\to\infty} \int_a^b |\, f_i(t)\,|^q \,\mathrm{d}t = |\, m(\cdot)\,|^q_{L^q(a,b)}. \tag{2.19}$$

But clearly (2.17) and (2.19) imply, by [10, th. 16.28],

$$(\, f_i(\cdot)\,) \xrightarrow{L^q} m(\cdot) \ \text{ strongly.} \tag{2.20}$$

On the other hand, by (2.9),

$$2 \max\, F_i\left(\left[a_i^j, b_i^j \right] \right) = 2F_i\left(c_i^j \right) = 2\widehat{F}_i\left(c_i^j \right) = \int_{a_i^j}^{b_i^j} \left| \widehat{f}_i(t) \right| \mathrm{d}t$$

$$\begin{aligned} 2 \max\, F_i(\,[a,b]\,) &\le \sum_{j\in J_i} \int_{a_i^j}^{b_i^j} \left| \widehat{f}_i(t) \right| \mathrm{d}t \\ &= \int_a^b \left| \widehat{f}_i(t) \right| \mathrm{d}t \le \int_a^b |\, f(t)\,|\,\mathrm{d}t \end{aligned}$$

hence, by (2.15),

$$2\sum_{i=0}^{\infty} |\, F_i\,(\cdot)\,|_{L^\infty(a,b)} \le \sum_{i=0}^{\infty} \int_a^b \left| \widehat{f}_i\,(t) \right| \mathrm{d}t \le \int_a^b |\, f\,(t)\,|\, \mathrm{d}t,$$

which shows, by (2.10), that

$$\left(|\, F_i\,(\cdot)\,|_{L^\infty(a,b)} \right) \searrow 0. \tag{2.21}$$

In particular, this implies that

$$F\,(\cdot) = \sum_{i=0}^{\infty} \widehat{F}_i\,(\cdot) \text{ uniformly on } [a,b]\,. \tag{2.22}$$

Joining together the info given by (2.18) and (2.21) one gets $m\,(\cdot) = 0$ a.e.; which, plugged into (2.20), yields

$$(\, f_i\,(\cdot)\,) \xrightarrow{L^q} 0 \text{ strongly.} \tag{2.23}$$

(Notice that (2.23) could also have been proved using, instead of the constructive arguments (2.17) to (2.20), some nonconstructive compactness tricks like the ones appearing in remarks below.)

Therefore $(\, F_i\,(\cdot)\,)$ converges in $W_0^{1,q}\,(\,[a,b]\,)$ to zero:

$$(\, F_i\,(\cdot)\,) \searrow 0 \text{ in } W_0^{1,q}\,(\,[a,b]\,) \quad \& \quad (\,|\, f_i\,(\cdot)\,|\,) \searrow 0 \text{ a.e., and } \left(\widehat{f}_i\,(\cdot) \right) \longrightarrow 0 \text{ a.e.;}$$

hence, passing to the limit in (2.14) and (2.15), one obtains

$$f\,(t) = \sum_{i=0}^{\infty} \widehat{f}_i\,(t) \text{ a.e.,} \quad |\, f\,(\cdot)\,|^q_{L^q(a,b)} = \sum_{i=0}^{\infty} \left| \widehat{f}_i\,(\cdot) \right|^q_{L^q(a,b)}$$

$$\lim_{N\to\infty} \int_a^b \left| f\,(t) - \sum_{i=0}^{N} \widehat{f}_i\,(t) \right|^q \mathrm{d}t = \int_a^b \left| f\,(t) - \sum_{i=0}^{\infty} \widehat{f}_i\,(t) \right|^q \mathrm{d}t =$$

$$= \lim_{N\to\infty} \int_a^b |\, f_{N+1}\,(t)\,|^q\, \mathrm{d}t = 0. \tag{2.24}$$

This completes the proof of Theorem 1. □

Remark 1. (Alternative proofs of strong L^q-convergence, $1 \le q \le p < \infty$, of $(\, f_i\,(\cdot)\,)$ to zero)

In order to prove strong $L^q\,(a,b)$-convergence of $(\, f_i\,(\cdot)\,)$, we did prefer to stick to the constructive proof appearing in (2.17)–(2.20), thereby discarding our prior functional-analysis arguments which follow.

To begin with, by the Dunford-Pettis theorem, since $|\, f_i\,(\cdot)\,| \le |\, f\,(\cdot)\,|$ a.e. $\forall\, i$, $(\, f_i\,(\cdot)\,)$ has a subsequence $(\, f_{i_k}\,(\cdot)\,)$ converging weakly, in $L^1\,(a,b)$, to some $m\,(\cdot) \in L^1\,(a,b)$, in particular $(\, F_{i_k}\,(t)\,) \to \int_a^t m\,(\tau)\; \mathrm{d}\tau$ $\forall\, t$; but since $(\, F_i\,(\cdot)\,)$ converges uniformly to zero, by (2.21), it must be $m\,(\cdot) = 0$ a.e.; and since the same arguments apply to any subsequence of $(\, f_i\,(\cdot)\,)$, the whole

$$(\, f_i\,(\cdot)\,) \overset{L^1}{\rightharpoonup} 0 \text{ weakly.} \tag{2.25}$$

Notice that (2.25) implies, with $R := \bigcap_{i=0}^{\infty} R_i$,

$$\int_a^b f_i(\tau)\,\chi_{R\cap[a,t]}(\tau)\,\mathrm{d}\tau = \int_a^t f_i(\tau)\,\chi_R(\tau)\,\mathrm{d}\tau$$
$$= \int_a^t f(\tau)\,\chi_{R_i}(\tau)\,\chi_R(\tau)\,\mathrm{d}\tau = \int_a^t f(\tau)\,\chi_R(\tau)\,\mathrm{d}\tau \underset{i\to\infty}{\to} 0 \quad \forall\, t \in [a,b]\,.$$

Thus

$$\int_a^t f(\tau)\,\chi_R(\tau)\,\mathrm{d}\tau = 0 \quad \forall\, t \in [a,b] \quad \text{hence}$$
$$f(t)\,\chi_R(t) = 0 \quad \text{for a.e. } t \in [a,b]\,,$$

so that

$$|\,R\,| = 0. \tag{2.26}$$

Moreover, since (see (2.12)),

$$\lim_{i\to\infty} \chi_{R_i}(t) = \chi_R(t) \quad \text{for a.e. } t, \tag{2.27}$$

we have, using Lebesgue dominated convergence,

$$0 = |\,R\,| = \int_a^b \chi_R(t)\,\mathrm{d}t = \int_a^b \lim_{i\to\infty} \chi_{R_i}(t)\,\mathrm{d}t$$
$$= \lim_{i\to\infty} \int_a^b \chi_{R_i}(t)\,\mathrm{d}t = \lim_{i\to\infty} |\,R_i\,|\,. \tag{2.28}$$

To reach (2.23) from (2.25) — instead of reaching (2.23) directly, avoiding (2.25), as in (2.17) to (2.20) — we have found three nonconstructive paths:

(a) First proof — by the Vitali convergence theorem (see e.g. [6, th. B.101]). Since $m(\cdot) = 0$ a.e., by (2.17),

$$(\,f_i(t)\,) \to 0 \quad \text{for a.e. } t.$$

Therefore, by the Egorov theorem,

$$(\,f_i(\cdot)\,) \text{ converges in measure to } 0.$$

Moreover, $\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ such that

$$\int_D |\,f_i(t)\,|^q\,\mathrm{d}t \le \int_D |\,f(t)\,|^q\,\mathrm{d}t \le \varepsilon \tag{2.29}$$

$\forall\, i \in \{0,1,2,\dots\}$ and for every measurable set $D \subset [a,b]$ with $|\,D\,| \le \delta$.

(b) Second proof — by Lebesgue dominated convergence. Using (2.17), (2.27) & (2.26) we obtain, due to (2.13),

$$\lim_{i\to\infty} \int_a^b |\,f_i(t)\,|^q\,\mathrm{d}t = \int_a^b \lim_{i\to\infty} (\,|\,f(t)\,|^q\,\chi_{R_i}(t)\,)\,\mathrm{d}t$$
$$= \int_a^b |\,f(t)\,|^q\,\chi_R(t)\,\mathrm{d}t = \int_R |\,f(t)\,|^q\,\mathrm{d}t = 0.$$

(c) Third proof — by the Riesz–Kolmogorov characterization of strong relative $L^q(a,b)$-compactness (see e.g. [4, Cor. 2.3.9]). Clearly it suffices to establish strong L^q-convergence of a subsequence $(f_{i_k}(\cdot))$. Since, for any $s<t$ in (a,b) and any i,

$$\left|\int_s^t f_i(\tau)\,\mathrm{d}\tau\right| \le \int_s^t |f(\tau)|\,\chi_{R_i}(\tau)\,\mathrm{d}\tau \le |f(\cdot)|_{L^1(a,b)},$$

we just need to show that

$$\int_a^{b-\delta} |f_i(t+\delta)-f_i(t)|^q\,\mathrm{d}t \longrightarrow 0 \text{ as } \delta\searrow 0, \text{ uniformly in } i. \tag{2.30}$$

To begin the proof of (2.30), by [4, Cor. 2.3.8],

$$\begin{aligned}&\forall\,\varepsilon>0\;\;\forall\,i\in\{0,1,2,\dots\}\;\;\exists\,\delta_i>0:\\ &0<\delta<\delta_i \;\Rightarrow\; \textstyle\int_a^{b-\delta}|f_i(t+\delta)-f_i(t)|^q\,\mathrm{d}t<\varepsilon.\end{aligned} \tag{2.31}$$

On the other hand, since $(|R_i|)\to 0$ (see (2.28)), uniform absolute continuity of the integral relative to Lebesgue measure implies, for the $\varepsilon>0$ fixed in (2.31), that

$$\exists\,i_\varepsilon\in\mathbb{N}: \int_a^b |f(t)|^q\,\chi_{R_i}(t)\,\mathrm{d}t<\varepsilon/2^{q+1} \quad \forall\,i>i_\varepsilon; \tag{2.32}$$

yielding

$$\begin{aligned}&\textstyle\int_a^{b-\delta}|f_i(t+\delta)-f_i(t)|^q\,\mathrm{d}t\\ &=\textstyle\int_a^{b-\delta}|f(t+\delta)\,\chi_{R_i}(t+\delta)-f(t)\,\chi_{R_i}(t)|^q\,\mathrm{d}t\\ &\le 2^q\left(\textstyle\int_a^{b-\delta}|f(t+\delta)|^q\,\chi_{R_i}(t+\delta)\,\mathrm{d}t+\int_a^{b-\delta}|f(t)|^q\,\chi_{R_i}(t)\,\mathrm{d}t\right)\\ &\le 2^{q+1}\textstyle\int_a^b|f(t)|^q\,\chi_{R_i}(t)\,\mathrm{d}t<\varepsilon \quad \forall\,i>i_\varepsilon.\end{aligned}$$

Finally, using (2.31) & (2.32) to set $\overline{\delta}:=\min\{\delta_0,\delta_1,\dots,\delta_{i_\varepsilon}\}$, one reaches our desired aim (2.30):

$$0<\delta<\overline{\delta} \quad\Rightarrow\quad \int_a^{b-\delta}|f_i(t+\delta)-f_i(t)|^q\,\mathrm{d}t<\varepsilon \quad \forall\,i,$$

thus again proving (2.23).

Remark 2. In case $F(\cdot)\in W_0^{1,p}([a,b])$, $1<p<\infty$, we can also prove strong L^q-convergence, $1<q\le p<\infty$, as well as strong L^1-convergence, of $(f_i(\cdot))$ to zero as follows.

Since

$$\sup_i |f_i(\cdot)|_{L^q(a,b)} \le |f(\cdot)|_{L^q(a,b)}<\infty$$

and, by (2.21),

$$\lim_{i\to\infty}\int_a^t f_i(\tau)\,\mathrm{d}\tau=0 \quad \forall\,t\in[a,b],$$

we have (see [3, ex. 2.50])

$$(f_i(\cdot)) \overset{L^q}{\rightharpoonup} 0 \text{ weakly.}$$

Therefore, as above ((2.25) to (2.28)), we can prove that

$$\lim_{i\to\infty} |R_i| = 0$$

hence

$$\lim_{i\to\infty}\int_a^b |f_i(t)|^q \,\mathrm{d}t = \lim_{i\to\infty}\int_a^b |f(t)|^q \chi_{R_i}(t)\,\mathrm{d}t$$
$$= \lim_{i\to\infty}\int_{R_i} |f(t)|^q \,\mathrm{d}t = 0,$$

which proves that

$$(f_i(\cdot)) \xrightarrow{L^q} 0 \text{ strongly.}$$

Let q_H be the Hölder conjugate exponent of q. By Hölder's inequality,

$$\int_a^b |f_i(t)|\,\mathrm{d}t \le \left(\int_a^b \mathrm{d}t\right)^{\frac{1}{q_H}} \left(\int_a^b |f_i(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}} \to 0 \text{ as } i\to\infty.$$

Thus

$$(f_i(\cdot)) \xrightarrow{L^1} 0 \text{ strongly.}$$

Remark 3. (Alternative proof of strong L^1-convergence of $(f_i(\cdot))$ to zero)

To prove that $(f_i(\cdot)) \xrightarrow{L^1} 0$ strongly we can also use Visintin's theorem [9, th. 1]. Indeed, since, due to (2.13),

$$0 \in ext\, K(t) \text{ for a.e. } t\in[a,b],$$
$$K(t) := \overline{co}\{0,\ f_0(t),\ f_1(t),\ \dots\},$$

and $(f_i(\cdot)) \overset{L^1}{\rightharpoonup} 0$ weakly, one must have (2.23).

Remark 4. (Alternative proof of uniform convergence of $s_k(\cdot) := \sum_{i=0}^k \widehat{F}_i(\cdot)$ to $F(\cdot)$)

To begin with, notice that the sequence $(F_i(\cdot))$ is equibounded:

$$|F_i(t)| \le \max F([a,b]) \quad \forall t\in[a,b] \quad \forall i\in\{0,1,2,\dots\}.$$

On the other hand, since $(f_i(\cdot))$ is equiintegrable ((2.29) holds true with $q=1$), $(F_i(\cdot))$ is also equicontinuous. Therefore, by Ascoli-Arzelà theorem, there exist a subsequence $(F_{i_k}(\cdot))$ and a continuous $h:[a,b]\to\mathbb{R}$ such that

$$(F_{i_k}(\cdot)) \to h(\cdot) \text{ uniformly on } [a,b],$$

in particular

$$(F_{i_k}(t)) \to h(t) \quad \forall t\in[a,b].$$

Since, for each t, the sequence $i\mapsto F_i(t)$ decreases — and any monotone sequence containing a convergent subsequence is itself convergent to the same limit — we have

$$(F_i(t)) \to h(t) \quad \forall t\in[a,b].$$

Thus, by Dini's theorem,

$$(F_i(\cdot)) \to h(\cdot) \ \text{ uniformly on } \ [a,b]. \tag{2.33}$$

On the other hand, since

$$s_k(\cdot) := \sum_{i=0}^{k} \widehat{F}_i(\cdot) = F(\cdot) - F_{k+1}(\cdot),$$

by (2.33) we conclude that

$$s_k(\cdot) \text{ converges uniformly to } \ (F - h)(\cdot) \ \text{ on } \ [a,b], \tag{2.34}$$

in particular

$$\left(\left| \widehat{F}_i(\cdot) \right|_{L^\infty(a,b)} \right) \searrow 0.$$

By (2.10),

$$|F_i(\cdot)|_{L^\infty(a,b)} = \left| \widehat{F}_i(\cdot) \right|_{L^\infty(a,b)}.$$

Then, by (2.33) and (2.34),

$$\left(|F_i(\cdot)|_{L^\infty(a,b)} \right) \searrow 0$$
$$\left(|s_k(\cdot) - F(\cdot)|_{L^\infty(a,b)} \right) \searrow 0,$$

thus proving (2.22) (again — and by a different method).

Remark 5. In case

$$F(\cdot) \in W^{1,\infty}([a,b]), \quad F(\cdot) \geq 0, \quad F(a) = 0 = F(b) \quad \& \quad F(\cdot) \not\equiv 0,$$

besides (2.3) & (2.4) to hold true for every $1 \leq q < \infty$, we also have

$$\left(\sum_{i=0}^{k} \widehat{f}_i(\cdot) \right) \overset{*}{\rightharpoonup} f(\cdot) \ \text{ in } \ L^\infty(a,b).$$

Indeed,

$$(f_i(\cdot)) \overset{*}{\rightharpoonup} 0 \ \text{ in } \ L^\infty(a,b)$$

due to Lebesgue dominated convergence theorem, since

$$(f_i(t)) \to 0 \ \text{ for a.e. } t$$

and

$$|f_i(\cdot)|_{L^\infty(a,b)} = |f(\cdot)\,\chi_{R_i}(\cdot)|_{L^\infty(a,b)} \leq |f(\cdot)|_{L^\infty(a,b)} \qquad \forall i \in \{0, 1, 2, \ldots\}.$$

Moreover,

$$\sum_{i=0}^{\infty} \widehat{f}_i(\cdot) = f(\cdot) \ \text{ in } \ L^\infty(a,b)$$

if and only if

$$\left(\operatorname*{ess\,sup}_{R_i} |f(\cdot)| \right) \to 0 \ \text{ as } \ i \to \infty.$$

Remark 6. (Decomposition of any scalar AC function $F(\cdot)$ into cap-components and cup-components)

Our cap-decomposition — appearing in (2.3) for special AC functions, namely those $F(\cdot) \geq 0$ as in (2.1) — can easily be extended to any function $F(\cdot) \in W^{1,p}([a,b])$, $1 \leq p < \infty$, as follows. Define

$$M_F(t) := F(a) + \frac{t-a}{b-a}\,[F(b) - F(a)] \quad \& \\ F^0(\cdot) := F(\cdot) - M_F(\cdot), \tag{2.35}$$

so that

$$F(\cdot) = M_F(\cdot) + \frac{F^0(\cdot)}{|F^0(\cdot)|}\,\left|F^0(\cdot)\right| \chi_{\mathcal{O}_0}(\cdot) \quad \& \\ F^0(a) = 0 = F^0(b), \tag{2.36}$$

where $\mathcal{O}_0$ is defined as in (2.2) with $F(\cdot)$ replaced by $\left|F^0(\cdot)\right|$. (More precisely, (2.36) is intended as saying that $F(\cdot) = M_F(\cdot)$ on $[a,b] \setminus \mathcal{O}_0$.)

Then we obtain a decomposition into cap components and cup components — recall (1.12) — for any AC function $F(\cdot)$, as above, just by applying (2.3) to $\left|F^0(\cdot)\right|$. Indeed,

$$F(\cdot) = M_F(\cdot) + \sum_{i=0}^{\infty} \frac{F^0(\cdot)}{|F^0(\cdot)|}\,\widehat{F}_i(\cdot)\,\chi_{\mathcal{O}_0}(\cdot) \quad \text{in } W^{1,p}([a,b]), \text{ hence uniformly,}$$

with each $\widehat{F}_i(\cdot)$ — defined as in (2.5)–(2.8) but starting, instead, from $F_0(\cdot) := \left|F^0(\cdot)\right|$ in order to respect (2.1) — a countably-piecewise-cap function. Together with (2.14), (2.13), (2.28) and Remark 7, this proves (1.9) to (1.11).

Remark 7. (Decomposition of any vectorial AC function $F(\cdot)$)

Considering now vectorial functions $F(\cdot) \in W^{1,p}([a,b], \mathbb{R}^m)$, with each coordinate $F^\nu(\cdot)$ satisfying (2.1), after determining the series $\sum_{i=0}^{\infty} \widehat{f^1}_i(\cdot)$ (see (2.4)) for the derivative $f^1(\cdot)$ of the first coordinate $F^1(\cdot)$ of $F(\cdot)$, one may determine the series $\sum_{i=0}^{\infty} \widehat{f^2}_i(\cdot)$ for the derivative $f^2(\cdot)$ of the second coordinate $F^2(\cdot)$ of $F(\cdot)$ and so on and so forth up to $f^m(\cdot)$. Thus (2.4) will represent not only a scalar equality in $L^p(a,b)$ but also a vectorial equality in $L^p((a,b), \mathbb{R}^m)$, for the vectorial derivative $f(\cdot)$ of $F(\cdot)$; and each coordinate $\widehat{f^\nu}_i$ of each $\widehat{f}_i = \left(\widehat{f^1}_i, \widehat{f^2}_i, \ldots, \widehat{f^m}_i\right)$ will satisfy (1.3) and (1.4).

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