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An Extension of the 1-Dim Lebesgue Integral of a Product of Two Functions

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Abstract: In this paper, our main aim is to present a reasonable extension of the 1-dim Lebesgue integral of the product of two functions, in case this Lebesgue integral does not exist (i.e., the integrals of its negative and positive parts are both ∞). This extension works fine quite generally, as shown by several examples, and it is based on general hypotheses guaranteeing the sign of the integral (in the sense of being necessarily <0 or $=0$ or else >0), without computing its actual value. For this purpose, our method provides much more precise results than the Lebesgue–Stieltjes integration by parts.

Keywords: extension of the 1-dim Lebesgue integral of a product; integral inequalities; Lebesgue–Stieltjes integration by parts

MSC: 26A42; 26A46; 26A48

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1. Introduction

Consider the class $\mathcal{F}^+$ of those functions

$$f(\cdot) \in L^1(a\,,b) \;\text{ having }\; \int_a^b f(t)\,dt = 0 \;\text{ and }\; \int_a^t f(\tau)\,d\tau \geq 0 \quad \forall\, t \in [a\,,b]\,,$$

i.e., each $f : (a\,,b) \subset \mathbb{R} \to [-\infty, \infty]$ $(a < b)$ is a Lebesgue-integrable function with zero-average and positive primitive. In our paper [1], we have constructed, for each such $f(\cdot) \in \mathcal{F}^+$, corresponding functions $\widehat{f}_i(\cdot) \in \mathcal{F}^+$ $(i = 0, 1, 2, \dots)$, whose sequence of partial sums converges a.e. and strongly in $L^1(a\,,b)$ to $f(\cdot)$, namely

$$f(\cdot) = \sum_{i=0}^{\infty} \widehat{f}_i(\cdot) \;\text{ a.e. and }\; \int_a^b \left| f(t) - \sum_{i=0}^{k} \widehat{f}_i(t) \right| dt \xrightarrow[k\to\infty]{} 0\,; \tag{1}$$

such that, for each i there exists a pair of *disjoint* open sets

$$A_i^+ \;\text{ where }\; \widehat{f}_i(\cdot) \geq 0 \;\text{ a.e. and }\; A_i^- \;\text{ where }\; \widehat{f}_i(\cdot) \leq 0 \;\text{ a.e.,}$$

while $\widehat{f}_i(\cdot) = 0$ a.e. on $[a\,,b] \setminus A_i^+ \setminus A_i^-$.

Another way of seeing (1) is through cap functions. We say that

$$F(\cdot) \geq 0 \;\text{ in }\; W_0^{1,1}([a\,,b]) \;\text{ is a }\; \textit{cap function} \;\text{ if }\; \left\{ \begin{array}{l} \exists\, c \in (a\,,b) : \\ F(\cdot) \;\text{ increases along }\; (a\,,c) \\ \text{and }\; F(\cdot) \;\text{ decreases along }\; (c\,,b) \end{array} \right\}. \tag{2}$$

Then, considering the primitives

$$F(t) := \int_a^t f(\tau)\, d\tau \quad \text{and} \quad \overset{\frown}{F}_i(t) := \int_a^t \overset{\frown}{f}_i(\tau)\, d\tau, \tag{3}$$

clearly

$$F(\cdot) \geq 0 \ \text{and}\ \overset{\frown}{F}_i(\cdot) \geq 0 \ \text{are in}\ W_0^{1,1}(\,[a\,,b]\,)\,; \tag{4}$$

and (3) with (1) yields what we call a *cap-decomposition* of such $F(\cdot) \geq 0$:

$$F(\cdot) = \sum_{i=0}^{\infty} \overset{\frown}{F}_i(\cdot) \quad \text{in}\ W_0^{1,1}(\,[a\,,b]\,)\,, \tag{5}$$

since our explicit construction in [1], from $F(\cdot) \geq 0$, of those functions $\overset{\frown}{f}_i(\cdot)$ in (1) is such that each $\overset{\frown}{F}_i(\cdot)$ is a *countably-piecewise-cap* function, in the sense that

$$\text{each}\ \overset{\frown}{F}_i(\cdot) \left\{ \begin{array}{l} \text{restricted to each connected} \\ \text{component of the open set} \\ \left\{ t \in (a\,,b)\,:\ \overset{\frown}{F}_i(t) > 0 \right\} \end{array} \right\} \text{is a cap function.} \tag{6}$$

Similarly to (2), we say that any

$$\begin{array}{l} F(\cdot) \leq 0 \ \text{in}\ W_0^{1,1}(\,[a\,,b]\,) \ \text{is a}\ \ \textit{cup function} \ \ \text{if} \\ -F(\cdot)\ \ \text{is a cap function.} \end{array} \tag{7}$$

Using these notations, more generally than in (5), we have constructed in [1] for any $F(\cdot)$ in $W_0^{1,1}(\,[a\,,b]\,)$, not necessarily ≥ 0, its *cap-cup-decomposition*, in the sense that

$$F(\cdot) = \sum_{i=0}^{\infty} \frac{F(\cdot)}{|F(\cdot)|}\, \overset{\frown}{F}_i(\cdot)\, \chi_{\mathcal{O}_0}(\cdot) \quad \text{in}\ W_0^{1,1}(\,[a\,,b]\,)\,, \tag{8}$$

where $\mathcal{O}_0 := \{\, t \in (a\,,b) : \quad F(t) \neq 0\,\}$, $\chi_E(\cdot)$ is the characteristic function of a set E (equal to 1 on E and to 0 elsewhere), each $\dfrac{F(\cdot)}{|F(\cdot)|}\, \overset{\frown}{F}_i(\cdot)$ is a *countably-piecewise-cap-cup* function and each $\overset{\frown}{F}_i(\cdot)$ is constructed (as in [1]) starting from $|F(\cdot)|$, instead of from $F(\cdot)$ (as mentioned above, after (5), for $F(\cdot) \geq 0$). As usual, we assume $-\infty\,.\,0 \,:=\, 0 \,=:\, \infty\,.\,0\,$.

In terms of derivatives, we obtain, using as before

$$f(\cdot) := F'(\cdot) \quad \text{and} \quad \overset{\frown}{f}_i(\cdot) := \overset{\frown}{F}{}_i'(\cdot)\,,$$

$$f(t) = \sum_{i=0}^{\infty} \frac{F(t)}{|F(t)|}\, \overset{\frown}{f}_i(t)\, \chi_{\mathcal{O}_0}(t) \quad \text{a.e. and in}\ L^1(a\,,b)\,. \tag{9}$$

The main aim of this paper is to use such tool (9) to obtain a reasonable extension of the definition of the Lebesgue integral

$$\int_a^b g(t)\, f(t)\, dt\,, \tag{10}$$

for $f(\cdot), g(\cdot) \in L^1(a\,,b)$ (but $g(\cdot)$, more generally, could be as in (15) below), which often works well when this integral does not exist in the Lebesgue sense (i.e., the integrals of its negative and positive parts are both ∞). This appears in Section 3 below.

However, we begin with Section 2, dedicated to apply the above mentioned tools (8) and (9) to determine the sign of the Lebesgue integral (10) or, equivalently, of

$$\int_a^b g(t)\, F'(t)\, dt\,, \tag{11}$$

without computing its value. By determining its sign we mean, more precisely, to find out whether this integral is <0, $=0$ or >0.

This problem of finding the sign of such an integral arises in various branches of Mathematics, both Pure and Applied, in particular in our area of expertise, namely to solve nonconvex problems of optimal control, of calculus of variations or of differential inclusions.

More precisely, our motivation to solve this problem of the sign of (10) or (11) came from needs we felt in our own research, on generalisations of the Liapunov theorem on the convexity of the range of nonatomic vector measures, to allow imposing constraints. The Liapunov theorem is the main tool to solve nonconvex problems, in pure and applied mathematics.

Finally, in Section 3, we use the tools developed in Section 2 to obtain our new extension of the Lebesgue integral (10). Roughly speaking, it consists of the following. Instead of computing the Lebesgue integral (10), i.e., over the whole interval $[a\,,b]$, which may not exist, we begin by replacing $f(t)$ in (10) by the summation in (9); and define our extension of the Lebesgue integral to be the sum of the integrals

$$\int_a^b g(t)\,\frac{F(t)}{|F(t)|}\,\widehat{f}_i(t)\,\chi_{\mathcal{O}_0}(t)\,dt\,, \tag{12}$$

computing this integral as follows. We compute first the Lebesgue integral of this integrand along each one of the connected components of the open set $\left\{\, t \in (a\,,b): \quad \widehat{F}_i(t) > 0 \,\right\}$; and then add those Lebesgue integrals to obtain (12). Notice that along each such a connected component, $\widehat{f}_i(t) = \widehat{F}_i'(t)$ necessarily changes sign, since $\widehat{F}_i(t)$ is zero at both extremities; thus, it is much more probable for such a Lebesgue integral to exist along each such connected components than along the whole interval $[a\,,b]$. (This is well exemplified below, in (62).) On the other hand, it is crucial to take into consideration that these functions $\widehat{f}_i(\cdot)$, $\widehat{F}_i(\cdot)$ (in particular, the above-mentioned connected components) are explicitly constructed, in [1], in a very natural way, so that our extension of Lebesgue integral is also quite natural.

We conclude Section 3 by presenting several illustrative examples demonstrating how this extension of the Lebesgue integral works fine, very generally, to overcome nonexistence. We also show that our extension works much better, in general, than the Lebesgue–Stieltjes integration by parts in order to determine the sign of the integral (11).

Clearly, our method may be applied to the Lebesgue integration along any kind of Lebesgue-measurable subset of $\mathbb{R}$. Indeed, where $F(\cdot) = 0$ we also have $F'(\cdot) = 0$ a.e., hence the Lebesgue integral of $g(\cdot)\,F'(\cdot)$ is zero there; while the set where $F(\cdot)$ is different from zero is an open set, hence a countable union of disjoint open intervals, a case which we have treated in full generality when all of these intervals are bounded. To consider the integration along an unbounded interval, one should partition such interval into a countable union of disjoint bounded subintervals, and apply our method to each one of them.

In case our extension and also any other reasonable extension of the Lebesgue integral both work well for a specific pair of functions $g(\cdot)$ and $F(\cdot)$ then—recalling that the value of any kind of reasonable integral of $g(\cdot)\,F'(\cdot)$ ultimately means the "signed area under the graph" of the integrand—clearly both integrals will yield the same value, in particular the same sign; thus, it is up to the user to pick the integral which is easier to handle. Obviously, it is useful to compute the above "area" for a wider class of integrands.

2. To Reach New Integral Inequalities through Our Explicitly Constructed Decompositions

Let

$$F(\cdot) \ \text{in} \ W_0^{1,1}(\,[a\,,b]\,) \ \text{be nonconstant.} \tag{13}$$

We begin by decomposing the nonempty open set

$$\mathcal{O}_0 := \{\, t \in (a\,,b) : \ F(t) \neq 0 \,\} = \bigcup_{j \in J_0} \left(a_0^j\,,b_0^j\right), \ J_0 \subset \mathbb{N}\,, \tag{14}$$

into its connected components $\left(a_0^j\,,b_0^j\right)$. On the other hand, we assume

$$\begin{array}{l} g(\cdot) \ \text{ to be in the class } \ \mathcal{M}_F \ \text{ of those } \ g : (a\,,b) \to [-\infty\,,\infty] \ \text{ whose} \\ \text{restriction to } \ \mathcal{O}_0\,, \ g|_{\mathcal{O}_0}(\cdot), \ \text{ is Lebesgue measurable} \\ \text{with values a.e. finite;} \end{array} \tag{15}$$

and say that

$$\begin{array}{l} g(\cdot) \ \text{ increases in } \ \mathcal{D} \subset \mathcal{O}_0 \ \text{ if} \\ \text{it equals a.e. a function (again denoted } \ g(\cdot)) \\ \text{satisfying } \ \text{“}\, t_1 < t_2 \ \textit{in} \ \mathcal{D} \ \Rightarrow \ -\infty < g(t_1) \leq g(t_2) < \infty \,\text{”}. \end{array} \tag{16}$$

We call $g(\cdot)$ *trivial* relative to $F(\cdot)$ whenever $g(\cdot)$ is constant along each connected component of the open set $\mathcal{O}_0$ in (14). More precisely, to each function $F(\cdot)$ as in (13) and (14) and to each sequence $g_0 = \left(g_0^j\right)_{j \in J_0}$ of real numbers, let us associate the function $g_0^F(\cdot)$ defined by taking a.e. in $\left(a_0^j\,,b_0^j\right)$ the constant value g_0^j and being 0 elsewhere, i.e.,

$$g_0^F(\cdot) := \sum_{j \in J_0} g_0^j \cdot \chi_{\left(a_0^j\,,b_0^j\right)}(\cdot) \ \text{ a.e. in } \ \mathcal{O}_0\,. \tag{17}$$

Then, we clearly have, for any function such as (17),

$$\int_a^b g_0^F(t)\, F'(t)\, dt = 0 \tag{18}$$

whenever this integral exists in the Lebesgue sense—and it certainly does exist when the sequence $g_0 = \left(g_0^j\right)_{j \in J_0}$ is bounded, regardless of how large is the Lebesgue measure $|\,\partial\,\mathcal{O}_0\,|$ of the boundary $\partial\,\mathcal{O}_0$ of the open set $\mathcal{O}_0$. Therefore, it becomes convenient to introduce a new definition:

$$\begin{array}{l} \text{given a nonconstant } \ F(\cdot) \ \text{ in } \ W_0^{1,1}(\,[a\,,b]\,), \\ \text{we say that a } \ g(\cdot) \in \mathcal{M}_F \ \text{ (as in (15)) is } \ \textit{nontrivial relative to} \ F(\cdot) \\ \text{whenever there is no sequence } \ g_0 \subset \mathbb{R} \ \text{ for which } \ g|_{\mathcal{O}_0}(\cdot) \ \text{ is as in (17).} \end{array} \tag{19}$$

(Notice that with $g(\cdot)$ of the form (17), the nonexistence in the Lebesgue sense is a real possibility, as shown below, in (60) and (61), using a sequence $g_0 \subset (0,\infty)$ and $F(\cdot) \geq 0$. However, what matters to us—at this point—is that in case $g(\cdot)$ is trivial relative to $F(\cdot)$—meaning that it does have the form (17) for some real sequence g_0—then the Lebesgue integral of $(\,g\,F'\,)(\cdot)$ is trivial, in the sense that its value is zero whenever it exists. On the other hand, as we shall see below—in (54) and Corollary 2—our new extension of the definition of Lebesgue integral is such that (18) always holds true.)

Here is our first result on the sign of the integral (11):

Theorem 1 (Sharp integral inequality)**.** *Let*

$$F(\cdot) \geq 0 \;\; \textit{be as in} \;\; (13) \;\; \textit{and} \;\; (14) \tag{20}$$

$$g(\cdot) \in \mathcal{M}_F \;\; \textit{be nontrivial relative to} \;\; F(\cdot) \;\; \textit{(i.e., as in (19))} \tag{21}$$

$$\textit{and have} \;\; g|_{\left(a_0^j,\, b_0^j\right)}(\cdot) \;\; \textit{increasing} \;\; \forall\, j \in J_0\,. \tag{22}$$

Then

$$-\infty \;\; \leq \;\; \int_a^b g(t)\, F'(t)\, dt \;\; < \;\; 0 \tag{23}$$

whenever this integral exists in the Lebesgue sense.

Moreover, this (Lebesgue) integral (23) exists whenever:
either

$$(\, g\, F'\,)^+(\cdot) \in L^1(a\,,b)\,;$$

or

$$(\, g\, F'\,)^-(\cdot) \in L^1(a\,,b) \quad \textit{hence} \;\; -\infty < \int_a^b g(t)\, F'(t)\, dt < 0\,;$$

or else

$$g(\cdot) \;\; \textit{in} \;\; (21) \;\; \textit{increases in} \;\; \mathcal{O}_0 \;\; \textit{and} \;\; \exists\, a', b' \in \mathcal{O}_0 \;\; \textit{such that:} \tag{24}$$

$$\textit{either} \;\; F|_{(a\,,a')}(\cdot) \;\; \textit{increases} \;\; \textit{and} \;\; F|_{(b',\,b)}(\cdot) \;\; \textit{decreases;} \tag{25}$$

$$\textit{or, more generally,} \;\; g\,(\,F'\,)^-(\cdot) \in L^1(a\,,a') \;\; \textit{and} \;\; g\,(\,F'\,)^+(\cdot) \in L^1(b',b)\,. \tag{26}$$

In our next (and more general) result, we allow $F(\cdot)$ to change sign:

Corollary 1 (General integral inequality)**.** *Assume* (13) *and* (21)*. Instead of* (22)*, suppose*

$$\frac{g\,F}{|F|}(\cdot) \;\; \textit{increases along each connected component of} \;\; \mathcal{O}_0\,. \tag{27}$$

Then

$$\int_a^b g(t)\, F'(t)\, dt \;\; \in \;\; [-\infty\,, 0) \tag{28}$$

whenever this integral exists in the Lebesgue sense;

$$\begin{array}{l} \textit{and one may replace, in} \;\; (27), \;\; \textit{``increases'' by ``decreases''} \\ \textit{provided one replaces, in} \;\; (28), \;\; [-\infty\,,0) \;\; \textit{by} \;\; (0\,,\infty]\,. \end{array} \tag{29}$$

Here is an elementary example: with $g(t) := \sin t$ and $F(t) := \cos t$, the integral of $(\,g\,F'\,)(\cdot)$ along $[\,-\pi/2\,,(2k-1)\,\pi/2\,]$, $k \in \mathbb{N}$, is <0, by application of Corollary 1. Of course, such a conclusion is obvious in this simple case, since, more precisely, its value is $-\,k\,\pi/2$, as one easily checks; however, our result explains geometrically why this integral is <0, and in what measure one may perturb $g(\cdot)$ and $F(\cdot)$ without the integral becoming zero or > 0.

Proof of Corollary 1. Let the integral in (28) exist. Since one may replace $g(\cdot)$ and $F(\cdot)$ by $g_1(\cdot) := g(\cdot)\,\dfrac{F(\cdot)}{|F(\cdot)|}\,\chi_{\mathcal{O}_0}(\cdot)$ and $F_1(\cdot) := |F(\cdot)|$ respectively, we can apply Theorem 1 either to $g_1(\cdot)$ and $F_1(\cdot)$ or else to $-\,g_1(\cdot)$ and $F_1(\cdot)$, thus proving (28) and (29). □

Proof of Theorem 1. Since we have already explicitly constructed, in [1], a cap-decomposition for $F(\cdot)$, as in (4)–(6), we claim first that the integral in (23) exists and is ≤ 0 whenever one replaces $F(\cdot)$ by a cap function $\widehat{F}(\cdot)$. In our second claim, we prove (23) when its integral exists. Finally, the third claim is dedicated to prove such existence under (26), in particular under (25).

Before starting this proof, let us recall some details on how our cap-decomposition is explicitly constructed in [1]. One defines, recursively,

$$F_{i+1}(\cdot) := F_i(\cdot) - \widehat{F}_i(\cdot), \text{ for } i = 0, 1, 2, \dots , \text{ with } F_0(\cdot) := F(\cdot) , \tag{30}$$

where $\widehat{F}_i(\cdot)$, the union of the disjoint *cap components* of $F_i(\cdot)$, is given as follows. For those i with

$$\mathcal{O}_i := \{ t \in (a\,,b) : F_i(t) \neq 0 \} \tag{31}$$

empty, we set $\widehat{F}_i(\cdot) := F_i(\cdot) = 0$; while for the other values of i we set

$$\widehat{F}_i(t) := \begin{cases} \min F_i\Big(\big[t\,,c_i^j\big]\Big) & \text{for } t \in \big[a_i^j,c_i^j\big] \\ \min F_i\Big(\big[c_i^j,t\big]\Big) & \text{for } t \in \big[c_i^j,b_i^j\big] \\ 0 & \text{for } t \in [a\,,b] \setminus \mathcal{O}_i\,, \end{cases} \tag{32}$$

with

$$\begin{aligned} &\Big(a_i^j,b_i^j\Big),\ j \in J_i\,,\ \text{the maximal nonempty intervals of } \mathcal{O}_i\,, \\ &\text{while } c_i^j := \min\Big\{ t \in \big[a_i^j,b_i^j\big] : F_i(t) = \max F_i\Big(\big[a_i^j,b_i^j\big]\Big)\Big\}. \end{aligned} \tag{33}$$

It is useful to define the set of relevant i's:

$$I := \{ i \in \{0, 1, 2, \dots\} : \mathcal{O}_i \neq \emptyset \} , \tag{34}$$

and the set of regular points of $\widehat{F}_i(\cdot)$:

$$\widehat{R}_i := \Big\{ t \in \mathcal{O}_i : \exists\, \widehat{F}_i'(t) \neq 0 \Big\} ,$$

so that

$$\widehat{F}_i'(t) = F'(t)\,\chi_{\widehat{R}_i}(t) \text{ a.e.} .$$

After these preliminaries, we finally start the proof by

Claim 1:

$$\exists \int_\alpha^\beta g(t)\,\widehat{F}'(t)\,dt \;\in\; [-\infty\,,0] , \tag{35}$$

$\widehat{F}(\cdot)$ (resp. $(\alpha\,,\beta)$) being any one of the $\widehat{F}_i(\cdot)$ (resp. $\Big(a_i^j,b_i^j\Big)$, $j \in J_i$) for $i \in I$.

Indeed, clearly the integral in (35) exists and has value zero whenever $g(\cdot)$ is constant along $(\alpha\,,\beta)$; thus, (using (22)) we now demonstrate that

$$g(\cdot) \text{ increasing nonconstant along } (\alpha\,,\beta) \;\Rightarrow\; \exists \int_\alpha^\beta g(t)\,\widehat{F}'(t)\,dt \;\in\; [-\infty\,,0) . \tag{36}$$

In fact, considering the interval

$$[\gamma\,,\delta] := \Big\{ t \in [\alpha\,,\beta] : \widehat{F}(t) = \max \widehat{F}([\alpha\,,\beta]) \Big\}$$

then clearly

$$\overset{\frown}{F}(\cdot) \text{ increases on } [\alpha\,,\gamma], \text{ is constant on } [\gamma\,,\delta] \text{ and decreases on } [\delta\,,\beta]\,.$$

Moreover,

$$\overset{\frown}{F}(\cdot) > 0 \text{ in } (\alpha\,,\beta)\,. \tag{37}$$

Using a lower (resp. upper) semicontinuous (see, e.g., [2]) increasing function $g_-(\cdot)$ (resp. $g_+(\cdot)$) equal to $g(\cdot)$ a.e., we define:

$$\gamma' := \min\{\, t \in [\alpha\,,\gamma] : \; g_+(t) = g_+(\gamma)\,\}$$
$$\delta' := \max\{\, t \in [\delta\,,\beta] : \; g_-(t) = g_-(\delta)\,\}$$

$$\widetilde{g}(t) := \begin{cases} g(t) - g_+(\gamma) & \text{if } \alpha < \gamma' \text{ or } \delta' = \beta \\ g(t) - g_-(\delta) & \text{if } \alpha = \gamma' \text{ and } \delta' < \beta\,, \end{cases} \qquad t \in (\alpha\,,\beta)\,.$$

In case $\alpha < \gamma'$ clearly

$$\widetilde{g}(t) \leq 0 \text{ and } \overset{\frown}{F}{}'(t) \geq 0 \text{ a.e. on } (\alpha\,,\gamma] \text{ hence } \widetilde{g}(t)\,\overset{\frown}{F}{}'(t) \leq 0 \text{ a.e. on } (\alpha\,,\gamma]$$
$$\widetilde{g}(t)\,\overset{\frown}{F}{}'(t) = 0 \text{ in } (\gamma\,,\delta)$$
$$\widetilde{g}(t) \geq 0 \text{ and } \overset{\frown}{F}{}'(t) \leq 0 \text{ a.e. on } [\delta\,,\beta) \text{ hence } \widetilde{g}(t)\,\overset{\frown}{F}{}'(t) \leq 0 \text{ a.e. on } [\delta\,,\beta)\,;$$

$$\widetilde{g}(\cdot) < 0 \text{ a.e. and } \overset{\frown}{F}(\cdot) \text{ increasing nonconstant, } \quad \text{along } (\alpha\,,\gamma')$$

since $\overset{\frown}{F}{}'(t) = 0$ a.e. on $[\alpha\,,\gamma'\,]$ would contradict the inequality in (37). Therefore

$$\int_\alpha^\beta \left(\widetilde{g}\,\overset{\frown}{F}{}'\right)^+(t)\,dt \;=\; 0 \quad \text{and} \quad \exists \int_\alpha^\beta \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt \;<\; 0\,,$$

even if $\left(g\,\overset{\frown}{F}{}'\right)(\cdot) \notin L^1(\alpha\,,\beta)$, due to:

$$\begin{aligned} \textstyle\int_\alpha^\beta \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt \;&=\; \textstyle\int_\alpha^{\gamma'} \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt + \int_{\gamma'}^\beta \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt \leq \\ &\leq\; \textstyle\int_\alpha^{\gamma'} \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt \;<\; 0\,. \end{aligned}$$

The case $\delta' < \beta$ is similar; while in case $\alpha = \gamma'$ and $\delta' = \beta$, clearly

$$\exists \int_\alpha^\beta \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt = -\overset{\frown}{F}(\delta)[\,g_-(\delta) - g_+(\gamma)\,] \;<\; 0\,.$$

By well-known properties of the Lebesgue integral (see, e.g., (Theorem 9.14 in [3])), we have thus shown that:

$$g(\cdot) \text{ is nonconstant along } (\alpha\,,\beta) \;\Rightarrow$$
$$\Rightarrow\; \exists \textstyle\int_\alpha^\beta g(t)\,\overset{\frown}{F}{}'(t)\,dt \;=\; \int_\alpha^\beta \widetilde{g}(t)\,\overset{\frown}{F}{}'(t)\,dt \;<\; 0\,.$$

This proves (36), hence (35).

Claim 2:

We now claim that, under (20)–(22), the existence of the integral in (23) implies

$$\int_a^b g(t)\,\widehat{F}_0'(t)\,dt < 0 \quad \text{and} \quad \int_a^b g(t)\,\widehat{F}_i'(t)\,dt \le 0 \quad \forall\, i \in I \tag{38}$$

hence the strict inequality in (23).

Indeed, consider first the *rhs* of (38). Concentrating attention on the intervals $\left(a_i^j, b_i^j\right)$, $j \in J_i$, of $\mathcal{O}_i$ in (31) where $\widehat{F}_i(\cdot) > 0$, one concludes that

$$\exists \int_{a_i^j}^{b_i^j} g(t)\,\widehat{F}_i'(t)\,dt \le 0 \;\; \forall\, j \in J_i \;\text{ and }\; \forall\, i \in I\,, \tag{39}$$

as in (35). Therefore, since the integral in (23) is assumed to exist,

$$\begin{aligned} \exists \int_a^b g(t)\,\widehat{F}_i'(t)\,dt &= \int_a^b g(t)\,F'(t)\,\chi_{\widehat{R}_i}(t)\,dt = \\ &= \sum_{j\in J_i} \int_{a_i^j}^{b_i^j} g(t)\,\widehat{F}_i'(t)\,dt \le 0 \;\; \forall\, i \in I\,. \end{aligned} \tag{40}$$

On the other hand, to prove the *lhs* of (38): clearly $\mathcal{O}_0 \neq \emptyset$ (by (13) and (14)); while, by (21),

$$\exists\, j_0 \in J_0: \;\text{ setting }\; a_0 := a_0^{j_0} \;\text{ and }\; b_0 := b_0^{j_0} \;\text{ then }\; g|_{(a_0,b_0)}(\cdot) \;\text{ is nonconstant;} \tag{41}$$

so that, by (39) with $i = 0$ and (36),

$$\exists \int_{a_0^j}^{b_0^j} g(t)\,\widehat{F}_0'(t)\,dt \le 0 \;\; \forall\, j \in J_0 \quad \text{and} \quad \int_{a_0}^{b_0} g(t)\,\widehat{F}_0'(t)\,dt < 0\,,$$

thus yielding (by (40)) the *lhs* of (38).

Finally, since (see (1) and (3))

$$g(t)\,F'(t) = g(t)\,\widehat{F}_0'(t) + \sum_{i=1}^{\infty} g(t)\,\widehat{F}_i'(t) \;\text{ a.e.}\,, \tag{42}$$

by (38) and using linearity and countable additivity of the integral, we have

$$\begin{aligned} &\exists \int_a^b g(t)\,F'(t)\,dt = \int_a^b g(t)\,\widehat{F}_0'(t)\,dt + \int_a^b \sum_{i=1}^{\infty} g(t)\,\widehat{F}_i'(t)\,dt = \\ &= \int_a^b g(t)\,\widehat{F}_0'(t)\,dt + \int_a^b g(t)\,F'(t) \sum_{i=1}^{\infty} \chi_{\widehat{R}_i}(t)\,dt = \\ &= \int_a^b g(t)\,\widehat{F}_0'(t)\,dt + \sum_{i=1}^{\infty} \int_a^b g(t)\,F'(t)\,\chi_{\widehat{R}_i}(t)\,dt = \\ &= \int_a^b g(t)\,\widehat{F}_0'(t)\,dt + \sum_{i=1}^{\infty} \int_a^b g(t)\,\widehat{F}_i'(t)\,dt \le \\ &\le \int_a^b g(t)\,\widehat{F}_0'(t)\,dt < 0\,. \end{aligned} \tag{43}$$

This completes the proof of (23) whenever its integral exists.

Finally, we conclude the proof of this theorem by proving

Claim 3:

$$(24) \text{ and } (26) \text{ imply existence of the integral in } (23). \tag{44}$$

Indeed, clearly (see (15) and (16)) we may assume, in (24) and (26),

$$a' < b' \quad \text{and} \quad -\infty < g(a') \leq g(b') < \infty .$$

Define

$$\begin{aligned} h_-(t) &:= \begin{cases} g(t) - g(a') & \text{for } t \in (a, a'] \\ 0 & \text{for } t \in [a', b) \end{cases} \\ h(t) &:= \begin{cases} g(a') & \text{for } t \in (a, a'] \\ g(t) & \text{for } t \in [a', b'] \\ g(b') & \text{for } t \in [b', b) \end{cases} \\ h_+(t) &:= \begin{cases} 0 & \text{for } t \in (a, b'] \\ g(t) - g(b') & \text{for } t \in [b', b) . \end{cases} \end{aligned}$$

Then

$$h_-|_{\mathcal{O}_0}(\cdot) \leq 0 \leq h_+|_{\mathcal{O}_0}(\cdot) \quad \text{and} \quad h_-(\cdot) + h(\cdot) + h_+(\cdot) = g(\cdot) \tag{45}$$

$$h_-(\cdot), \; h(\cdot) \text{ and } h_+(\cdot) \text{ all increase in } \mathcal{O}_0 \tag{46}$$

$$h(\cdot) \in L^\infty(\mathcal{O}_0) . \tag{47}$$

Moreover, using (41) and decreasing a', increasing b' (if necessary), we may assume that

$$h|_{(a_0, b_0)}(\cdot) \quad \text{is nonconstant.} \tag{48}$$

Similarly to (40), by (35) and (46) (with $g(\cdot)$ replaced by $h_-(\cdot)$ and $h_+(\cdot)$),

$$\begin{aligned} &\exists \int_a^b h_-(t)\, F'(t)\, dt \;\Rightarrow \\ &\Rightarrow\; \exists \int_a^b h_-(t)\, \widehat{F}_i'(t)\, dt = \int_a^b h_-(t)\, F'(t)\, \chi_{\widehat{R}_i}(t)\, dt \leq 0 \;\; \forall i \in I \end{aligned} \tag{49}$$

$$\begin{aligned} &\exists \int_a^b h_+(t)\, F'(t)\, dt \;\Rightarrow \\ &\Rightarrow\; \exists \int_a^b h_+(t)\, \widehat{F}_i'(t)\, dt = \int_a^b h_+(t)\, F'(t)\, \chi_{\widehat{R}_i}(t)\, dt \leq 0 \;\; \forall i \in I ; \end{aligned} \tag{50}$$

while, by (36), (with $g(\cdot)$ replaced by $h(\cdot)$),

$$\begin{aligned} &(47),\, (46) \text{ and } (48) \;\Rightarrow \\ &\Rightarrow\; \exists \int_a^b h(t)\, \widehat{F}_i'(t)\, dt \leq 0 \;\; \forall i \in I \quad \text{and} \quad \int_a^b h(t)\, \widehat{F}_0'(t)\, dt < 0 . \end{aligned} \tag{51}$$

Integrating now along $[a, b]$ the equality (recall the analogous (42))

$$h(t)\, F'(t) = h(t)\, \widehat{F}_0'(t) + \sum_{i=1}^{\infty} h(t)\, \widehat{F}_i'(t) \;\text{ a.e. in }\; (a, b)$$

and using (51) we conclude, as in (43), that

$$\exists \int_a^b h(t)\, F'(t)\, dt \leq \int_a^b h(t)\, \widehat{F}_0'(t)\, dt < 0 . \tag{52}$$

Moreover, since $\exists\, F'(t) = 0$ a.e. in $(a\,,b) \setminus \mathcal{O}_0$ and $|\,h_-(t)\,| = -h_-(t) = g(a') - g(t)$ on $(a\,,a'] \cap \mathcal{O}_0$, we have:

$$\begin{aligned}
&\textstyle\int_a^b (\,h_-\,F'\,)^+(t)\,dt = \int_a^{a'} (\,-F'\,|h_-|\,)^+(t)\,dt = \\
&= \textstyle\int_a^{a'} |\,h_-(t)\,|\,(\,F'\,)^-(t)\,dt = \\
&= g(a')\textstyle\int_a^{a'} (\,F'\,)^-(t)\,dt - \int_a^{a'} g(t)\,(\,F'\,)^-(t)\,dt \le \\
&\le |\,g(a')\,|\,\Big|\,(\,F'\,)^-(\cdot)\,\Big|_{L^1(a\,,a')} + \Big|\,g(\cdot)\,(\,F'\,)^-(\cdot)\,\Big|_{L^1(a\,,a')} < \infty\,,
\end{aligned}$$

by (26); and since, similarly, $h_+(t) = g(t) - g(b') \ge 0$ on $[b', b) \cap \mathcal{O}_0$, we also have:

$$\begin{aligned}
&\textstyle\int_a^b (\,h_+\,F'\,)^+(t)\,dt = \int_{b'}^b h_+(t)\,(\,F'\,)^+(t)\,dt = \\
&= \textstyle\int_{b'}^b g(t)\,(\,F'\,)^+(t)\,dt - g(b')\int_{b'}^b (\,F'\,)^+(t)\,dt \le \\
&\le \Big|\,g(\cdot)\,(\,F'\,)^+(\cdot)\,\Big|_{L^1(b',\,b)} + |\,g(b')\,|\,\Big|(\,F'\,)^+(\cdot)\,\Big|_{L^1(b',\,b)} < \infty\,,
\end{aligned}$$

again by (26). Therefore

$$(26) \quad \Rightarrow \quad \int_a^b (\,h_-\,F'\,)^+(t)\,dt < \infty \quad \text{and} \quad \int_a^b (\,h_+\,F'\,)^+(t)\,dt < \infty\,;$$

so that

$$\exists \int_a^b h_-(t)\,F'(t)\,dt \in [-\infty\,,\infty) \;\; \text{and} \;\; \exists \int_a^b h_+(t)\,F'(t)\,dt \in [-\infty\,,\infty)\,. \tag{53}$$

Thus, by (49), (50), (52) and (53),

$$\exists\, \textstyle\int_a^b (\,h\,F'\,)(t)\,dt < 0 \quad \text{and}$$

$$\begin{aligned}
&\exists\, \textstyle\int_a^b (\,h_-\,F'\,)(t)\,dt + \int_a^b (\,h_+\,F'\,)(t)\,dt = \\
&= \textstyle\int_a^b \sum\limits_{i=0}^{\infty} h_-(t)\,\widehat{F}_i'(t)\,dt + \int_a^b \sum\limits_{i=0}^{\infty} h_+(t)\,\widehat{F}_i'(t)\,dt = \\
&= \sum\limits_{i=0}^{\infty} \textstyle\int_a^b h_-(t)\,\widehat{F}_i'(t)\,dt + \sum\limits_{i=0}^{\infty} \int_a^b h_+(t)\,\widehat{F}_i'(t)\,dt \le 0
\end{aligned}$$

hence, by (Theorem 9.14 in [3]) together with (45),

$$\begin{aligned}
&\exists\, \textstyle\int_a^b g(t)\,F'(t)\,dt = \\
&= \textstyle\int_a^b h(t)\,F'(t)\,dt + \int_a^b h_-(t)\,F'(t)\,dt + \int_a^b h_+(t)\,F'(t)\,dt < 0\,.
\end{aligned}$$

Notice finally that, thanks to the extra hypothesis (26),

$$\begin{aligned}
&(\,h_-\,F'\,)(\cdot) \notin L^1(a\,,b) \quad \text{or} \quad (\,h_+\,F'\,)(\cdot) \notin L^1(a\,,b) \quad \Rightarrow \\
&\Rightarrow \quad \exists\, \textstyle\int_a^b g(t)\,F'(t)\,dt = -\infty \;\; < \;\; 0\,.
\end{aligned}$$

This completes the proof of (44), and hence of Theorem 1. □

3. To Extend Reasonably the Definition of Lebesgue Integral of a Product of Two Functions

The main aim of this section is to propose—in (54) and (64)—a reasonable extension of the definition of Lebesgue integral of a product whenever the integrals of the negative and positive parts of the integrand both have ∞ value. After presenting its definition together with illustrating examples, we turn in Remark 1 to demonstrate why our method works much better than the Lebesgue–Stieltjes integration by parts to determine whether the integral of the product has value =0, <0 or >0.

As explained above—recall (12)—our extension of the Lebesgue integral (11) consists, roughly speaking, on a term-by-term Lebesgue integration of the series obtained on replacing $F(\cdot)$ in the integrand by its series (5):

$$\text{under } (13) \text{ and } (15)\,, \quad \int_a^b g(t)\,F'(t)\,dt \;:=\; \lim_{M\to\infty}\sum_{\substack{i\in I\\ i\le M}} \lim_{N\to\infty}\sum_{\substack{j\in J_i\\ j\le N}} \int_{a_i^j}^{b_i^j} g(t)\,\frac{F(t)}{|F(t)|}\,\widehat{F}_i'(t)\,dt\,, \tag{54}$$

whenever both these limits exist (and they may not exist, as in (67) below), considering each $\widehat{F}_i(\cdot)$ constructed as in (30)–(33) but starting instead from $F_0(\cdot) := |F(\cdot)|$, so that the *rhs* of (54) indeed represents term-by-term integration of the cap-cup-decomposition of $F(\cdot)$, as defined after (7). Notice that our new integral (54) may be well-defined with a finite value only if—for each fixed $i \in I$—the last integral in (54) goes to zero as $j \to \infty$ in case $J_i = \mathbb{N}$. Moreover, in case the monotony condition (27) (or its alternative applying (29)) is satisfied, then the integrals after the summations in (54) all exist, as true Lebesgue integrals (see (35)); while the *rhs* of (54) is well-defined, as the limit of a monotone sequence. Indeed, in case (27) holds true then this *rhs* will be the limit of a decreasing sequence $S_M \subset [-\infty\,,0]$ (or of an increasing sequence $S_M \subset [0\,,\infty]$, if (29) is used). Clearly, the *rhs* of (54) will have exactly the same value as the true Lebesgue integral whenever this one exists. Hence, one easily proves the following

Corollary 2 (Equality and inequality with generalized integral)**.** *For any $F(\cdot)$ as in (13) the integral (11) defined through (54) exists at least whenever:*

(a) Either $g(\cdot)$ is trivial relative to $F(\cdot)$, as in (17), in which case (18) holds true;
(b) Or $g(\cdot)$ satisfies (21) and (27) (resp. its modification by (29)), in which case (28) (resp. its modification by (29)) holds true.

Example 1 (Cap-decompositions successfully applied to a non-trivial case where Lebesgue integral does not exist)**.** *Here is a simple example of functions $g(\cdot)$ and $F(\cdot)$ not satisfying the extra hypothesis (26):*

$$g(t) := -\phi'(t)\,\phi(t)^{-\alpha}$$
$$F(t) := \phi(t)^{\beta+1}\,\sin^2\!\left(\frac{1}{\phi(t)}\right) \;\; \textit{and} \;\; 0<\beta\le\alpha<1\,,$$

$$\textit{where } \; \phi(t) := 1-|t|, \; \textit{so that:}$$
$$\phi'(t) = -t\,/\,|t| \;\; \textit{and} \;\; \phi(-1) = 0 = \phi(1)\,,$$

$$g(\cdot)\in L^1(-1\,,1) \;\; \textit{increases}$$
$$F(\cdot)\in W_0^{1,1}(\,[-1\,,1]\,) \;\; \textit{has} \;\; F(\cdot)>0 \;\; \textit{a.e.}$$

$$F'(t) = \left[\,(\beta+1)\,\phi'\,\phi^{\beta}\,\sin^2\!\left(\frac{1}{\phi}\right) - \phi'\,\phi^{\beta-1}\,\sin\!\left(\frac{2}{\phi}\right)\right](t) \;\; \textit{a.e.}$$

$$g(t)\,F'(t) = \left[\,-(\beta+1)\,\phi^{\beta-\alpha}\,\sin^2\!\left(\frac{1}{\phi}\right) + \phi^{\beta-\alpha-1}\,\sin\!\left(\frac{2}{\phi}\right)\right](t) \;\; \textit{a.e.}.$$

The improper Riemann integral of such $(\,g\,F'\,)(\cdot)$ equals, as one easily checks,

$$\int_{-1}^{1} g(t)\,F'(t)\,dt =$$
$$= \; -2\,(\beta+1)\int_1^{\infty} r^{\alpha-\beta-2}\,\sin^2 r\,dr + 2^{\beta-\alpha+1}\int_2^{\infty} r^{\alpha-\beta-1}\,\sin r\,dr$$

and has a finite value, as is well-known, since $\alpha-\beta-1 \in [-1\,,0)$.

More precisely, as follows from Theorem 1,

$$\exists \, improper \;\; Riemann \;\; \int_{-1}^{1} g(t)\, F'(t)\, dt \;\; \in \;\; (-\infty\,,0)\,; \tag{55}$$

while (26) is false and the corresponding Lebesgue integral does not exist since, for any $\varepsilon \in (0\,,1)$,

$$\begin{aligned} &\psi(\cdot) := \phi^{\beta-\alpha-1}\, \sin\!\left(\frac{2}{\phi}\right)(\cdot) \;\; \Rightarrow \\ &\Rightarrow \;\; \psi^{-}(\cdot) \;\; and \;\; \psi^{+}(\cdot) \notin L^1(-1\,,-1+\varepsilon) \cup L^1(1-\varepsilon\,,1)\,. \end{aligned}$$

Indeed, from Theorem 1, along each interval where $F(\cdot) > 0$ *the proper Riemann integral of* $(g\,F')(\cdot)$ *is* <0*, hence also the integral in (55) is* < 0 *along the whole interval* $(-1\,,1)$*. As one easily checks, in this case both limits in the rhs of definition (54) exist, yielding a total sum* < 0*, in accordance with the Riemann integration.*

Example 2 (Cap-decompositions successfully applied to a trivial case where Lebesgue integral does not exist)**.** *Consider in* $[a\,,b] = [0\,,1]$ *the points and intervals*

$$b_0^0 := 0\,, \quad b_0^j := \frac{b_0^{j-1}+1}{2} = \frac{2^j-1}{2^j}\,, \;\; for \;\; j = 1,\,2,\,3,\,\ldots$$

$$\begin{aligned} &E^j := \left(b_0^{j-1},\, b_0^j\right), \\ &E_+^j := \left(b_0^{j-1},\, b_0^{j-1} + \frac{1}{2^{j+1}}\right), \;\; E_-^j := \left(b_0^{j-1} + \frac{1}{2^{j+1}}\,,\, b_0^j\right). \end{aligned} \tag{56}$$

Set

$$f(t) := \begin{cases} 1 & at \; those \;\; t \in E_+^j\,, \;\; j = 1,\,2,\,3,\,\ldots \\ -1 & at \; those \;\; t \in E_-^j\,, \;\; j = 1,\,2,\,3,\,\ldots \end{cases} \tag{57}$$

and

$$F(t) := \int_a^t f(\tau)\, d\tau\,, \tag{58}$$

so that $F(\cdot)$ *satisfies (13) and (14) with* $\mathcal{O}_0 = \bigcup_{j=1}^{\infty} E^j$.

Consider also the sequence $g_0 = \left(g_0^j\right)_{j\in\mathbb{N}}$, $g_0^j := 2^j$*, and the corresponding function* $g_0^F(\cdot)$*, i.e., (as in (17))*

$$g_0^F(\cdot) := \sum_{j=1}^{\infty} 2^j.\chi_{E^j}(\cdot) \;\; a.e. \; in \;\; \mathcal{O}_0\,. \tag{59}$$

Then, one obtains

$$\begin{aligned} &\int_0^1 (g_0^F\, F')^{+}(t)\, dt = \int_0^1 \sum_{j=1}^{\infty} 2^j.\chi_{E_+^j}(t)\, dt = \\ &= \sum_{j=1}^{\infty} \int_{E_+^j} 2^j\, dt = \sum_{j=1}^{\infty} 2^j \left| E_+^j \right| = \sum_{j=1}^{\infty} \frac{1}{2} = \infty \end{aligned} \tag{60}$$

and, similarly,

$$\int_0^1 \left(g_0^F\, F'\right)^{-}(t)\, dt = \int_0^1 \sum_{j=1}^{\infty} 2^j\,.\chi_{E_-^j}(t)\, dt = \infty\,. \tag{61}$$

Therefore, the integral $\int_0^1 (g_0^F\, F')(t)\, dt$ *does not exist in the Lebesgue sense. However, it does exist—with value zero—according to our new definition (54):*

$$\begin{aligned}&\textstyle\int_0^1 (\, g_0^F\, F'\,)(t)dt = \sum\limits_{j=1}^{\infty} \int_{E^j} g_0^F(t)\, F'(t)\; dt = \\ &= \sum\limits_{j=1}^{\infty} \left(\int_{E_+^j} 2^j\; dt - \int_{E_-^j} 2^j\; dt \right) = 0\,.\end{aligned} \tag{62}$$

Our cap-cup-decomposition—appearing in (8) for those $F(\cdot)$ which are zero on the extremities a and b—can easily be extended to any function $F(\cdot) \in W^{1,1}(\,[a\,,b]\,)$ as in [1]. Namely, define

$$M_F(t) := F(a) + \frac{t-a}{b-a}\,[\,F(b) - F(a)\,] \quad \text{and} \quad F^0(\cdot) := F(\cdot) - M_F(\cdot)$$

yielding:

$$F(\cdot) = M_F(\cdot) + F^0(\cdot) \quad \text{and} \quad F^0(a) = 0 = F^0(b)\,. \tag{63}$$

Thus, one may obtain still another extension of our extension (54), as follows:

$$\int_a^b g(t)\, F'(t)\; dt := \frac{F(b) - F(a)}{b-a} \int_a^b g(t)\; dt + \int_a^b g(t)\, F^{0\prime}(t)\; dt\,, \tag{64}$$

whenever everything is well-defined. Or, more precisely, in case $F(a) = F(b)$ we define the *rhs* of (64) to be just its second integral, taken in the sense of our definition (54), whenever its limit exists; while otherwise (64) only makes sense whenever the first integral on its *rhs* does exist in the Lebesgue sense, its second integral exists in our sense (54) and, finally, their sum is well-defined (i.e., unless $\infty - \infty$ appears).

Example 3 (Our extension of Lebesgue integral exists even for wildly oscillating integrands with $g(\cdot)$ satisfying one of the monotony conditions appearing in (27) or (29))**.** *Let* $[a\,,b] := [0\,,1]$*;* E^j*,* E_+^j*,* E_-^j *as in (56) and let* $F(\cdot)$ *be as in (58) with*

$$f(t) := \begin{cases} (-1)^{j+1} & \text{at those } t \in E_+^j\,,\ \ j = 1,2,3,\ldots \\ (-1)^j & \text{at those } t \in E_-^j\,,\ \ j = 1,2,3,\ldots\,. \end{cases} \tag{65}$$

Consider

$$g(t) := \sum_{j=1}^{\infty} (-\,1)^j . \frac{1}{1-t} . \chi_{E^j}(t)\,.$$

Then, the integral $\int_0^1 (\, g\, F'\,)(t)\; dt$ *does not exist in the Lebesgue sense. However, by Corollary 2 (b),*

$$\exists \int_a^b g(t)\, F'(t)\; dt = \sum_{j=1}^{\infty} \int_{E^j} (-\,1)^j . \frac{1}{1-t} . f(t)\; dt = \log\left(\frac{9}{8}\right) . \sum_{j=1}^{\infty} 1 = \infty\,.$$

Example 4 (Our extension of the Lebesgue integral may exist and have a finite value even for wildly oscillating integrands with $g(\cdot)$ not satisfying the monotony conditions appearing in (27) and (29))**.** *Let* $[a\,,b] := [0\,,1]$*;* E^j*,* E_+^j*,* E_-^j *as in (56) and let* $F(\cdot)$ *be as in (58) with* $f(\cdot)$ *as in (65). Then, considering*

$$g(t) := \sum_{j=1}^{\infty} 2^j . t . \chi_{E^j}(t)\,, \tag{66}$$

the Lebesgue integral of $(\, g\, F'\,)(\cdot)$ *does not exist, as one easily checks; while the limit in (54) does exist, so that our extension of the Lebesgue integral is well-defined in this case:*

$$\exists \quad \lim_{M\to\infty} \sum_{\substack{i\in I \\ i\le M}} \lim_{N\to\infty} \sum_{\substack{j\in J_i \\ j\le N}} \int_{a_i^j}^{b_i^j} g(t)\, \tfrac{F(t)}{|F(t)|}\, \widehat{F}_i^{\,\prime}(t)\, dt =$$
$$= \sum_{j\in J_0} \int_{E^j} g(t)\, \tfrac{F(t)}{|F(t)|}\, \tfrac{d}{dt}\, |F(t)|\, dt = \sum_{j=1}^{\infty} \int_{E^j} g(t)\, F'(t)\, dt = \sum_{j=1}^{\infty} \frac{(-1)^j}{2^{j+2}} = -\,\frac{1}{12}\,.$$

Notice that our extension (54) of the Lebesgue integral exists with a finite value in Examples 2 and 4, even with $g(\cdot) \notin L^1(a\,,b)$, thus justifying definition (15).

Example 5 (Our extension of Lebesgue integral does not work in some cases)**.** *Let* $[a,b] := [0\,,1]$. *Take* E^j, E_+^j, E_-^j, $f(\cdot)$ *and* $F(\cdot)$ *as in Example 4. Then, considering*

$$g(t) := \frac{1}{1-t}\,,$$

one obtains

$$\int_0^1 (\,g\,F'\,)^+(t)\; dt = \int_0^1 (\,g\,F'\,)^-(t)\; dt = \log 2\, \sum_{j=1}^{\infty} 1 = \infty\,,$$

so that the integral $\int_0^1 (\,g\,F'\,)(t)\; dt$ *does not exist in the Lebesgue sense. Moreover, this integral also does not exist in our generalized sense, because the sequence in the rhs of (54), while being bounded does not have a limit, continuing forever to oscillate between two constants:*

$$\begin{aligned} &\sum_{j=1}^{N} \int_{a_0^j}^{b_0^j} g(t)\, \tfrac{F(t)}{|F(t)|}\, \widehat{F}_0^{\,\prime}(t)\, dt = \sum_{j=1}^{N} \int_{E^j} g(t)\, F'(t)\, dt = \\ &= \sum_{j=1}^{N} (-1)^j \log\!\left(\tfrac{9}{8}\right) = \log\!\left(\tfrac{9}{8}\right) . \tfrac{(-1)^N - 1}{2}\,. \end{aligned} \tag{67}$$

On the other hand, it is easy to modify $g(\cdot)$ *in such a way that the sum of the first* $2N$ *terms of the series becomes* $2^{2N} - 1$, *while the sum of the first* $2N+1$ *terms of the series becomes* $-\,2^{(2N+1)} - 1$. *Thus, the sequence of partial sums of the series will have odd order terms near* $-\,4^N$ *and even order terms near* 4^N, *leading to oscillations with arbitrarily large amplitudes, tending to* ∞. *In contrast, one obtains oscillations with amplitude decreasing to zero on replacing, in (66),* $2^j.\,t$ *by* $j^{-1}.\,(1-t)^{-1}$; *case in which our extension of Lebesgue integral exists, with value* $-\log(2).\,\log(9/8)$; *while Lebesgue integral does not exist, since one ends up with an alternating-sign conditionally-convergent series which does not converge absolutely. Obviously, the usual elementary necessary condition applies here: the series for (54) may converge to a finite limit only if its general term approaches zero; which also becomes a sufficient condition in case we are dealing with alternating series whose general term decreases in the modulus to zero.*

In order to finalize this discussion, notice that our extensions (54) or (64), of the definition of the integral, are in a sense unique (whenever they exist). Indeed, someone might wonder whether, e.g., in (54), the last integral could possibly be defined along intervals different from the mentioned $\left(a_i^j, b_i^j\right)$. Consider, e.g., the integral in (62) and take intervals $\left(d_0^j, d_0^{j+1}\right) \subsetneq \left(b_0^{j-1}, b_0^{j+1}\right)$, e.g., with $d_0^j := c_0^j$, where $c_0^j = b_0^{j-1} + \dfrac{1}{2^{j+1}}$ is the max point defined in (32)–(34). In such case, one would consider in the *rhs* of (54) the integrals over the intervals: (b_0^1, d_0^1) and $\left(d_0^j, d_0^{j+1}\right)$ for $j \in \mathbb{N}$. Then, taking $d_0^j := b_0^{j-1} + \dfrac{\alpha}{2^j}$ with $\alpha \in (\,0\,,1/2\,]$, the value of this integral (62), with $g(\cdot)$ trivial relative to $F(\cdot)$, would be α, not zero. Therefore, this would not be a reasonable extension of the Lebesgue integral, since the integrals with integrand of the type (18)—i.e., with $g(\cdot)$ trivial relative to $F(\cdot)$—always have zero value whenever $(\,g\,F'\,)(\cdot) \in L^1(a\,,b)$; thus, in a reasonable extension of this Lebesgue integral, they should—for coherence sake—still have the value zero, regardless of existing or not in the Lebesgue sense. Together with Corollary 2 (a), this proves that the unique way of reaching the value zero for all the integrals with $g(\cdot)$ trivial relative to $F(\cdot)$ in $W_0^{1,1}([a,b])$ is by using the intervals $\left(a_i^j, b_i^j\right)$ as in (54).

Remark 1 (Cap-cup-decompositions yield sharper sign results than the Lebesgue–Stieltjes integration by parts)**.** *The point we wish to make in this remark is that our results (Theorem 1, Corollary 1 and the above definition (54), hence Corollary 2) often classifies the integral (11) sharply among the following qualitative classes:*

$$\begin{array}{ll} \nexists \textit{ integral}; & \exists \textit{ integral } < 0; \\ \exists \textit{ integral } = 0; & \exists \textit{ integral } > 0 \end{array} \tag{68}$$

in many cases for which the Lebesgue–Stieltjes integration by parts yields no information at all on this.

To see why, recall first that the Lebesgue-Stieltjes integration by parts (see, e.g., (Corollary 5.40 in [4]) or (p. 104 in [5])) applied to any $F(\cdot)$ in $W_0^{1,1}([a,b])$ and $g(\cdot)$ having $g'(\cdot) \in L^1(a,b)$ yields the equality

$$\int_a^b \left[\int_a^t g'(\tau)\, d\tau \right] F'(t)\, dt \;=\; \int_a^b -F(t)\, g'(t)\, dt\,. \tag{69}$$

Clearly, e.g., under (13), (21) and (27), the last integral always exists, in the Lebesgue sense, and:

$$\textit{unless } \; g'(\cdot) = 0 \; \textit{ a.e. in } \mathcal{O}_0\,, \; \textit{it has value } \; < 0\,.$$

On the other hand, also $\int_a^b g(t)\, F'(t)\, dt < 0$, by Corollary 2 (in case this integral exists in the Lebesgue sense or defined through (54)). The other case mentioned in (29) is similar. This yields a useful mnemonic rule: under these hypotheses (13), (21) and (27) (see also (29)) and assuming that $g'(\cdot) \in L^1(a,b)$ then, unless $g'(\cdot) = 0$ a.e. in $\mathcal{O}_0$, the integral (11) has the same sign as the integral of $(-F g')(\cdot)$; e.g., in case $F(\cdot) > 0$ and $g'(\cdot) > 0$ a.e. in $\mathcal{O}_0$, the integral of $(-F g')(\cdot)$ is < 0; hence, the same happens with the integral (11).

However, our main point here is that the equality (69) assumes the form $0 = 0$ whenever $g'(\cdot) = 0$ a.e. in $\mathcal{O}_0$, which happens, e.g., if: (a) $g(\cdot)$ is countably-piecewise-constant along $\mathcal{O}_0$, but different from the rhs of (17) along at least one interval $\left(a_0^{j_0}, b_0^{j_0}\right) \subset \mathcal{O}_0$, namely different from $g_0^{j_0}$ on a nonempty open subset; or (b) $g(\cdot)$ is, along each interval of $\mathcal{O}_0$, the nonconstant increasing Cantor function appearing, e.g., in (Example 1.43 in [4]); or (c) $g(\cdot)$ is, along each interval of $\mathcal{O}_0$, a strictly monotone singular function (see [6] and references therein for examples of such functions). Under under (13), (21) and (27) (see also (29)), this equality $0 = 0$ is void of information concerning the classification (68) for the integral in (11), while our Corollary 2 ensures that in all these cases the integral (11) is either <0 or >0.

Moreover, (69) does not help in the classification (68) more generally whenever $g(\cdot)$ is not the integral of its derivative, since it ignores what is the difference between $g(t)$ and $g(a)$ plus the integral up to t of $g'(\cdot)$; while this difficulty does not appear in our above results (Theorem 1, Corollaries 1 and 2).

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