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Geometric Characterization of Validity of the Lyapunov Convexity Theorem in the Plane for Two Controls under a Pointwise State Constraint

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Abstract: This paper concerns control BVPs, driven by ODEs $x'(t) = u(t)$, using controls $u^0(\cdot)$ & $u^1(\cdot)$ in $L^1((a,b),\mathbb{R}^2)$. We ask these two controls to satisfy a very simple restriction: at points where their first coordinates coincide, also their second coordinates must coincide; which allows one to write $(u^1 - u^0)(\cdot) = v(\cdot)(1, f(\cdot))$ for some $f(\cdot)$. Given a relaxed non bang-bang solution $\overline{x}(\cdot) \in W^{1,1}([a,b],\mathbb{R}^2)$, a question relevant to applications was first posed three decades ago by A. Cellina: does there exist a bang-bang solution $\widehat{x}(\cdot)$ having lower first-coordinate $\widehat{x}_1(\cdot) \leq \overline{x}_1(\cdot)$? Being the answer always yes in dimension $d = 1$, hence without $f(\cdot)$, as proved by Amar and Cellina, for $d = 2$ the problem is to find out which functions $f(\cdot)$ "are good", namely "allow such 1-lower bang-bang solution $\widehat{x}(\cdot)$ to exist". The aim of this paper is to characterize "goodness of $f(\cdot)$" geometrically, under "good data". We do it so well that a simple computational app in a smartphone allows one to easily determine whether an explicitly given $f(\cdot)$ is good. For example: non-monotonic functions tend to be good; while, on the contrary, strictly monotonic functions are never good.

Keywords: Lyapunov convexity theorem; pointwise state constraints; convexity of the range of vector measures; linear control BVPs; nonconvex linear ordinary differential inclusions

MSC: 28B05; 34A60

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Citation: Carlota, C.; Lopes, M.; Ornelas, A. Geometric Characterization of Validity of the Lyapunov Convexity Theorem in the Plane for Two Controls under a Pointwise State Constraint. *Axioms* **2024**, *13*, 611. https://doi.org/10.3390/axioms13090611

Academic Editors: Hyun Soo Chung, Loredana Ciurdariu and Eugenia Grecu

Received: 31 July 2024

Revised: 29 August 2024

Accepted: 2 September 2024

Published: 9 September 2024

1. Introduction

We aim at finding a simple, easily applicable in practice, way to determine whether an explicitly given pointwise-constrained control BVP (boundary value problem), using two controls in the plane $\mathbb{R}^2$, does possess bang-bang solutions. The famous Lyapunov theorem on the convexity of the range of vector measures is not valid here, without extra hypotheses, since bang-bang solutions do not always exist, due to our pointwise state-constraint.

Here are the main articles and books dealing with the original Lyapunov theorem, without pointwise state-constraints: [1–9].

The plan of the remaining of this introduction is as follows. We begin by describing—in its first Section 1.1—a new notation that has been a key tool to obtain our new results. Then—in Section 1.2—this powerful notation is used to present with precision not only previous results but also the aim of this paper.

1.1. Description of Our BVP and Its Reduction to a Functional Equation

We start this description by taking two controls, i.e., functions

$$u^0(\cdot) \,\&\, u^1(\cdot) \;\; in \;\; L^1\Big((a,b),\mathbb{R}^2\Big), \tag{1}$$

and defining the corresponding relaxed, or convexified, differential inclusion

$$x'(t) \in co\Big\{u^0(t), u^1(t)\Big\} \;\; a.e., \tag{2}$$

where $co\{a, b\}$ means the convex hull of the set $\{a, b\}$, namely the closed straight-line segment with extremities $a, b \in \mathbb{R}^2$, usually also denoted by $[a, b]$, together with its nonconvex, or bang-bang, counterpart

$$x'(t) \in \Big\{u^0(t), u^1(t)\Big\} \;\; a.e. \tag{3}$$

and the corresponding spaces of AC (absolutely continuous) solutions

$$\mathcal{X}^{co} := \Big\{x(\cdot) \in W^{1,1}\Big([a, b], \mathbb{R}^2\Big) \; satisfying \; (2)\Big\}$$

$$\mathcal{X}^{bb} := \Big\{x(\cdot) \in W^{1,1}\Big([a, b], \mathbb{R}^2\Big) \; satisfying \; (3)\Big\}.$$

Then we fix a relaxed non bang-bang solution

$$\overline{x}(\cdot) \in \mathcal{X}^{co} \backslash \mathcal{X}^{bb} \tag{4}$$

and define the space of those AC functions $x(\cdot)$ satisfying the same boundary conditions:

$$\overline{\mathcal{X}} := \overline{x}(\cdot) + W_0^{1,1}\Big([a, b], \mathbb{R}^2\Big); \tag{5}$$

together with its subspaces consisting of *relaxed* and of *bang-bang*, also called *true*, *solutions* to the control BVP defined by (3) & $\overline{x}(\cdot)$, respectively:

$$\overline{\mathcal{X}}^{co} := \overline{\mathcal{X}} \cap \mathcal{X}^{co} \qquad \& \qquad \overline{\mathcal{X}}^{bb} := \overline{\mathcal{X}} \cap \mathcal{X}^{bb}; \tag{6}$$

and the subspaces of solutions $x(\cdot)$ whose first-coordinate $x_1(\cdot)$ is *dominated* by $\overline{x}_1(\cdot)$, i.e.,

$$\overline{\mathcal{X}}^{co}_{-} := \Big\{x(\cdot) \in \overline{\mathcal{X}}^{co} : \; x_1(t) \leq \overline{x}_1(t) \; \forall t\Big\} \tag{7}$$

$$\overline{\mathcal{X}}^{bb}_{-} := \Big\{x(\cdot) \in \overline{\mathcal{X}}^{bb} : \; x_1(t) \leq \overline{x}_1(t) \; \forall t\Big\}. \tag{8}$$

Then, the main aim of this paper is to find hypotheses ensuring that:

$$\begin{aligned} &for\ the\ \ \overline{x}(\cdot)\ fixed\ in\ (4) \\ &there\ does\ exist\ a\ corresponding\ \ \widehat{x}(\cdot) \in \overline{\mathcal{X}}^{bb}_{-}. \end{aligned} \tag{9}$$

Notice that we exclude $\mathcal{X}^{bb}$ in (4) since for $\overline{x}(\cdot) \in \mathcal{X}^{bb}$ trivially $\widehat{x}(\cdot) := \overline{x}(\cdot) \in \overline{\mathcal{X}}^{bb}_{-}$.

First hypothesis: denoting by $(\ldots)_1$ & $(\ldots)_2$ the first & second coordinates in $\mathbb{R}^2$ and

$$\begin{aligned} &defining \;\; v(\cdot) := (u^1 - u^0)_1(\cdot), \\ &assume \;\; v(t) = 0 \;\; \Rightarrow \;\; (u^1 - u^0)_2(t) = 0, \\ &hence \;\; (u^1 - u^0)(\cdot) = v(\cdot)(1, f(\cdot)) \;\; in \;\; L^1\big((a, b), \mathbb{R}^2\big) \\ &for\ some \;\; f(\cdot) \;\; which,\ clearly,\ must\ satisfy \\ &f(t) = \frac{(u^1 - u^0)_2}{v}(t) \;\; a.e.\ where\ v(t) \neq 0. \end{aligned} \tag{10}$$

Let us now show how (10) can be used to greatly simplify our problem of ensuring (9). First step: since—by definition of convex hull—any $x(\cdot) \in \mathcal{X}^{co}$ must satisfy

$$\begin{aligned} &x'(t) = (1 - \lambda)(t)u^0(t) + \lambda(t)u^1(t) \;\; a.e.\ for\ some \\ &\lambda(\cdot) \in \Lambda := \{\lambda(\cdot) \in L^\infty(a, b) : \; \lambda(t) \in [0, 1] \; a.e.\} \end{aligned} \tag{11}$$

hence

$$\begin{aligned}&x(\cdot)\in\mathcal{X}^{co}\ \Leftrightarrow\\&\exists\,\lambda(\cdot)\in\Lambda:\ x'(t)=u(t):=u^0(t)+v(t)(1,f(t))\lambda(t)\ \ a.e.\ \ \&\\&x(t)=x(a)+\int_a^t u^0(\tau)d\tau+\int_a^t v(\tau)(1,f(\tau))\lambda(\tau)d\tau\ \ \forall t\in[a,b],\end{aligned}\tag{12}$$

then $\widehat{x}(\cdot)$ & $\overline{x}(\cdot)$ both satisfy (12) with $x(a)$ replaced by $\overline{x}(a)$ and $\lambda(\cdot)$ replaced by corresponding $\widehat{\lambda}(\cdot)$ & $\overline{\lambda}(\cdot)$ in Λ, hence the bang-bang solution $\widehat{x}(\cdot)\in\overline{\mathcal{X}}_-^{bb}$ we are looking for must satisfy

$$\widehat{x}(b)=\overline{x}(b)\ \Leftrightarrow\ \int_a^b v(t)(1,f(t))\left(\overline{\lambda}-\widehat{\lambda}\right)(t)dt=0.$$

Second step: a simpler way of seeing our BVP is through functions

$$\widehat{g}(\cdot)\in W_0^{1,1}([a,b]),\ \ \widehat{g}(t):=\int_a^t v(\tau)\left(\overline{\lambda}-\widehat{\lambda}\right)(\tau)d\tau.\tag{13}$$

Indeed, notice first that given one such $\widehat{g}(\cdot)$, we may always recover the corresponding $\widehat{\lambda}(\cdot)$:

$$\widehat{\lambda}(t)=\begin{cases}\left(\overline{\lambda}-\dfrac{\widehat{g}'}{v}\right)(t) & for\ a.e.\ t\ where\ v(t)\neq 0\\ 0 & elsewhere;\end{cases}\tag{14}$$

consequently the desired $\widehat{x}(\cdot)$ will be given by (12) with $\widehat{x}(a):=\overline{x}(a)$ and $\lambda(\cdot)$ replaced by $\widehat{\lambda}(\cdot)$ as in (14). On the other hand, clearly, it is intended in (9) that

$$(\overline{x}_1-\widehat{x}_1)(\cdot)\geq 0\ \ \&\ \ \widehat{x}(\cdot)\neq\overline{x}(\cdot)\ \ \&\ \ \widehat{\lambda}(t)\in\{0,1\}\ a.e.$$

which, through (10) and (12)–(14), is equivalent to

$$\widehat{g}(\cdot)\geq 0\ \ \&\ \ \widehat{g}(\cdot)\not\equiv 0\ \ \&\ \ \widehat{g}'(t)\in\{\gamma_-(t),\gamma_+(t)\}\ a.e.,$$

where

$$\gamma_-(\cdot):=-\left[\left(1-\overline{\lambda}\right).v\right](\cdot)\ \ \&\ \ \gamma_+(\cdot):=\left(\overline{\lambda}.v\right)(\cdot).\tag{15}$$

Third step: taking into consideration the second step, it turns out convenient to define the next spaces:

$$\begin{gathered}\mathcal{G}^{\geq}:=\left\{g(\cdot)\in W_0^{1,1}([a,b]):\ g(t)\geq 0\ \ \forall t\in[a,b]\ \ \&\ \ g(\cdot)\not\equiv 0\right\}\\ \mathcal{G}^{>}:=\left\{g(\cdot)\in\mathcal{G}^{\geq}:\ g(t)>0\ for\ a.e.\ t\in(a,b)\right\}\\ \widetilde{\mathcal{G}}:=\left\{g(\cdot)\in W_0^{1,1}([a,b]):\ g'(t)\in co\{\gamma_-(t),\gamma_+(t)\}\ a.e.\right\}\\ \widehat{\mathcal{G}}:=\left\{g(\cdot)\in\widetilde{\mathcal{G}}:\ g'(t)\in\{\gamma_-(t),\gamma_+(t)\}\ a.e.\right\}\\ \widetilde{\mathcal{G}}^{\geq}:=\mathcal{G}^{\geq}\cap\widetilde{\mathcal{G}}\ \ \&\ \ \widetilde{\mathcal{G}}^{>}:=\mathcal{G}^{>}\cap\widetilde{\mathcal{G}}\ \ \&\ \ \widehat{\mathcal{G}}^{\geq}:=\mathcal{G}^{\geq}\cap\widehat{\mathcal{G}}.\end{gathered}\tag{16}$$

Observe that $(f.g')(\cdot)$ must be in $L^1(a,b)$ for $g(\cdot)$ in each one of the spaces in the last three lines, due to (10) and (15).

Fourth step: using such spaces, here are the conclusions one should take from the above discussion: to find the desired bang-bang coefficient $\widehat{\lambda}(\cdot)$ yielding the aimed bang-bang control

$$\widehat{u}(\cdot):=u^0(\cdot)+v(\cdot)(1,f(\cdot))\widehat{\lambda}(\cdot)$$

—hence, through (12) with $x(a)$ & $\lambda(\cdot)$ replaced by $\overline{x}(a)$ & $\widehat{\lambda}(\cdot)$, respectively, obtain the sought-for bang-bang solution $\widehat{x}(\cdot)$ in $\overline{\mathcal{X}}_{-}^{bb}$—what one needs is to prove the existence of a solution $\widehat{g}(\cdot)$ to the next *bang-bang functional equation*:

$$\exists\, \widehat{g}(\cdot) \in \widehat{\mathcal{G}}^{\geq}: \quad \int_a^b f(t)\widehat{g}'(t)dt = 0. \tag{17}$$

Complicated as this may seem, we are at least lucky that part of it has already been treated. Indeed, we have shown in [10] (Theorem 3.2 & Remark 3.3) that the bang-bang part of (17) can be replaced by its relaxed counterpart with strict inequality. More precisely, in order to guarantee the existence of solution $\widehat{g}(\cdot)$ to the bang-bang functional Equation (17), it suffices to guarantee the existence of solution $\widetilde{g}(\cdot)$ to the corresponding *relaxed functional equation*:

$$\exists\, \widetilde{g}(\cdot) \in \widetilde{\mathcal{G}}^{>}: \quad \int_a^b f(t)\widetilde{g}'(t)dt = 0. \tag{18}$$

However, one may also consider a more general version of (18), which has the additional merit of being completely equivalent to (17). It is applicable namely in case one finds a $\widetilde{g}(\cdot)$ which happens to be already bang-bang on some closed subset of $[a,b]$, denoted by $[a,b]\backslash\widetilde{\mathcal{O}}$, with $\widetilde{\mathcal{O}}$ open nonempty in $[a,b]$. More precisely, here is our *generalized relaxed functional equation*:

$$\begin{aligned}
&\exists\, \widetilde{g}(\cdot) \in \widetilde{\mathcal{G}}^{\geq}: \quad \int_a^b f(t)\widetilde{g}'(t)dt = 0 \quad \& \\
&\exists\, open\ nonempty\ set\ \ \widetilde{\mathcal{O}} \subset (a,b): \\
&\widetilde{g}(t) > 0\ \ a.e.\ in\ \ \widetilde{\mathcal{O}} \quad \& \\
&\widetilde{g}'(t) \in \{\gamma_-(t), \gamma_+(t)\}\ \ a.e.\ on\ \ [a,b]\backslash\widetilde{\mathcal{O}}.
\end{aligned} \tag{19}$$

As shown in [10] (Theorem 3.2 & Remark 3.3),

$$(19) \qquad is\ equivalent\ to \qquad (17). \tag{20}$$

Finally, we complete this section by drawing attention to the fact that the convex differential inclusion defining the space $\widetilde{\mathcal{G}}$ in (16) can be ignored whenever our control problem has

$$\begin{aligned}
&\text{“}good\ data\text{''},\ i.e., \\
&v(\cdot)\ in\ (10)\ \ (which\ we\ may\ assume\ wlg\ \geq 0\ a.e.) \\
&stays\ away\ from\ 0; \\
&\&\ \ \overline{x}'(\cdot)\ \ (as\ in\ (11)\ with\ \lambda(\cdot)\ replaced\ by\ \overline{\lambda}(\cdot)) \\
&stays\ away\ from\ the\ extremal\ controls\ (1).
\end{aligned} \tag{21}$$

More precisely, from now on, we guarantee (21) by assuming that

$$\exists\, \overline{\varepsilon} \in \left(0, \frac{1}{2}\right): \quad \begin{array}{l} \overline{\varepsilon} \leq v(t) \\ \overline{\varepsilon} \leq \overline{\lambda}(t) \leq 1 - \overline{\varepsilon} \end{array} \quad a.e.. \tag{22}$$

Then any $\overline{g}(\cdot) \in W_0^{1,\infty}([a,b])$ with $\overline{g}(\cdot) \not\equiv 0$ will give us at once an element of the space $\widetilde{\mathcal{G}}$ in (16), namely a Lipschitz solution to its convex differential inclusion. Indeed, taking any such $\overline{g}(\cdot)$ and setting $\widetilde{g}(\cdot) := \overline{g}(\cdot).\overline{\varepsilon}^2/|\overline{g}'(\cdot)|_\infty$ one gets $\widetilde{g}'(t) \leq \overline{\varepsilon}^2 = \overline{\varepsilon}.\overline{\varepsilon} \leq \overline{\lambda}(t).v(t) =: \gamma_+(t)$ and $-\widetilde{g}'(t) \leq \overline{\varepsilon}^2 = \overline{\varepsilon}.\overline{\varepsilon} \leq [1 - \overline{\lambda}(t)].v(t) =: -\gamma_-(t)$ for *a.e.* t, i.e.,

$$\gamma_-(t) \leq -\overline{\varepsilon}^2 \leq \widetilde{g}'(t) \leq \overline{\varepsilon}^2 \leq \gamma_+(t)\ \ a.e.. \tag{23}$$

Notice also the following: (22) and (10) $\Rightarrow$ $f(\cdot) \in L^1(a,b)$.

Therefore, considering the *Lipschitz relaxed functional equation*

$$\exists\, \overline{g}(\cdot) \in W_0^{1,\infty}([a,b]) : \;\; \overline{g}(t) > 0 \;\; a.e. \;\;\; \& \;\; \int_a^b f(t)\overline{g}'(t)dt = 0, \tag{24}$$

we have shown in (23) and [10] (Theorem 3.2 & Remark 3.3) that

$$\begin{aligned} &(24)\ \&\ (21)\ \ (or,\ more\ precisely,\ (24)\ \&\ (22))\ \Rightarrow \\ &\Rightarrow\ (18)\ \Rightarrow\ (17)\ \Leftrightarrow\ \overline{\mathcal{X}}_-^{bb} \neq \emptyset. \end{aligned} \tag{25}$$

From now on, we will say that

$$\begin{aligned} &the\ control\ problem\ described\ in\ (4)\ to\ (9) \\ &has\ \text{“}good\ data\text{''}\ whenever\ (22)\ holds\ true. \end{aligned} \tag{26}$$

1.2. Previous Results Plus Motivation and Aim of This Paper

The pointwise-constrained control BVP (9) appeared for the first time in the paper [11], in dimension $d = 1$—hence without $f(\cdot)$ in (10)—case in which a solution always exists. More precisely, in this case a relaxed solution having minimal (respectively maximal) first-coordinates always exists and is automatically bang-bang, as proved in [12] (Theorem 1.1).

The paper [13] considered constant control-vertices in $\mathbb{R}^d$; while the papers [14,15] both deal only with scalar states and controls, and a description of their results can be seen in [10] (after (1.13)). Our paper [12] deals with an arbitrary dimension $d \in \mathbb{N}$ and an arbitrary number of integrable controls, while its pointwise state-constraint is determined by a constant direction, given by a unit-vector V, the special case $V = (1,0)$ yielding the first-coordinate. This paper was followed by [10], where we generalized the results of [12] by considering—instead of $x'(\cdot) = u(\cdot)$—the more general differential equation $x'(\cdot) + A(\cdot)x(\cdot) = u(\cdot)$, where $A(\cdot)$ is a $d \times d$-integrable matrix, with a moving AC directional vector $V(\cdot)$.

For the $d = 2$ case studied here, namely with $f(\cdot)$ as in (10) and under the pointwise state-constraint in (8), defining:

$$\begin{aligned} &\text{“}f(\cdot)\ is\ good\text{''}\ \text{— } for\ the\ \ \overline{x}(\cdot)\ we\ have\ fixed\ in\ (4) \text{ —} \\ &to\ mean\ that\ \ \overline{\mathcal{X}}_-^{bb} \neq \emptyset\ \ (see\ (8)) \end{aligned} \tag{27}$$

then

$$f(\cdot)\ \ constant\ is\ good\ \text{ — } for\ any\ \ \overline{x}(\cdot)\ as\ in\ (4) \text{ —} \tag{28}$$

as one easily checks by Lemma 3.4 in [11] (or Theorem 1.1 in [12]).

On the other hand, see [12] (2.10) & [16] (Theorem 1),

$$\begin{aligned} &f(\cdot)\ \ a.e.\ strictly\ monotonic \text{ — } e.g.,\ \ f(\cdot)\ \ affine\ nonconstant \text{ —} \\ &is\ not\ good,\ for\ any\ \ \overline{x}(\cdot)\ as\ in\ (4). \end{aligned} \tag{29}$$

The motivation for the research presented in this paper was the following. While we have proved in [10] (Theorem 3.2) a very useful theoretical result applicable in particular in this case $d = 2$ we are treating, namely the equivalence appearing above in (20), this result does not explain, unfortunately, what geometrical behavior of $f(\cdot)$ is good or not good. In particular, it does not answer simple geometrical questions like "Why $f(\cdot)$ affine nonconstant is never good?" or "Can a parabolic $f(\cdot)$ be good?". The aim of this paper is precisely to answer, in a simple and geometrically intuitive way, such questions.

In what follows, after proving our two main results—Theorems 1 and 2, stated in (38) and (40)—and after presenting several useful properties and conditions in Sections 3 and 4, we present in Section 5 several examples of successful application of our results to explicitly given functions $f(\cdot)$ depending on parameters. Namely, we have been able—using e.g., a simple computational and plotting app called MathStudio on a smartphone—to easily determine precise ranges of parameters making $f(\cdot)$ a good function, in the sense (27). In the last Section 6 we discuss the importance of the novel results in this paper.

2. Our Geometric Characterization Using Weighted Averages

Our new geometrical characterization is based on a class of *weights* $w(\cdot)$, through which we define and compute lateral weighted-averages of the function $f(\cdot)$ in (10), hence a new function $f_{ab}^{w}(\cdot)$ that will play a central role.

Indeed, we associate to each weight $w(\cdot)$, namely

$$w(\cdot) \in L^1(a,b) \quad having \quad w(t) \geq 0 \;\; a.e. \quad \& \quad \int_a^b w(t)dt = 1,$$

the two lateral indefinite integrals

$$W_a(t) := \int_a^t w(\tau)d\tau \quad \& \quad W_b(t) := 1 - W_a(t) = \int_t^b w(\tau)d\tau.$$

We also associate to each

$$\begin{array}{l} f(\cdot) \;\; as\; in\; (10)\; the\; corresponding\; class\; of\; weights \\ \mathcal{W}^f := \left\{ \begin{array}{ll} w(\cdot) \in L^1(a,b): & w(t) \geq 0 \; a.e. \\ & \& \; W_a((a,b)) = (0,1) \\ & \& \; (f.w)(\cdot) \in L^1(a,b) \end{array} \right\}, \\ assume \;\; \mathcal{W}^f \;\; nonempty \end{array} \tag{30}$$

and, for each $w(\cdot) \in \mathcal{W}^f$, define the corresponding *lateral weighted-averages*

$$\begin{array}{l} f_a^w(t) := \dfrac{1}{W_a(t)} \displaystyle\int_a^t f(\tau)w(\tau)d\tau - \overline{f}^w \;\; for\; t \in (a,b] \;\; \& \\ f_b^w(t) := \dfrac{1}{W_b(t)} \displaystyle\int_t^b f(\tau)w(\tau)d\tau - \overline{f}^w \;\; for\; t \in [a,b), \\ where \;\; \overline{f}^w := \displaystyle\int_a^b f(t)w(t)dt. \end{array} \tag{31}$$

Then

$$f_a^w(b) = 0 = f_b^w(a);$$

while zero becomes, along (a,b), a convex combination of $f_a^w(t)$ & $f_b^w(t)$ with nonzero coefficients:

$$0 = W_a(t)\, f_a^w(t) + (1 - W_a)(t)\, f_b^w(t) \;\; \forall t \in (a,b); \tag{32}$$

thus, setting

$$f_{ab}^w(t) := \frac{f_b^w(t) - f_a^w(t)}{b-a} \;\; for\; t \in (a,b) \tag{33}$$

one reaches the equivalences

$$\begin{array}{l} f_{ab}^w(t) = 0 \;\Leftrightarrow\; f_a^w(t) = f_b^w(t) \;\Leftrightarrow\; f_a^w(t) = 0 = f_b^w(t) \;\Leftrightarrow \\ \Leftrightarrow\; f_a^w(t).f_b^w(t) = 0, \; \forall t \in (a,b); \\ together\; with\; the\; inequality\; (due\; to\; (32)): \\ f_a^w(t).f_b^w(t) \leq 0 \;\; \forall t \in (a,b). \end{array} \tag{34}$$

Now, define the new class of *cap functions* $\widehat{g}(\cdot)$:

$$\widehat{\mathcal{G}} := \left\{ \begin{array}{ll} g(\cdot) \in W_0^{1,1}([a,b]): & g(t) > 0 \;\; \forall t \in (a,b) \\ & \& \; \exists\, m \in (a,b): (m-t)g'(t) \geq 0 \; a.e. \end{array} \right\}$$

obtaining, for $\widehat{g}(\cdot) \in \widehat{\mathcal{G}}$,

$$\begin{aligned} &\left|\widehat{g}'(\cdot)\right|_1 := \int_a^b \left|\widehat{g}'(t)\right| dt = 2M \ \& \\ &\left|\widehat{g}'(t)\right| = \begin{cases} \widehat{g}'(t) & a.e.\ in\ (a,m) \\ -\widehat{g}'(t) & a.e.\ in\ (m,b), \end{cases} \end{aligned} \tag{35}$$

where $M := \max\ \widehat{g}([a,b])$; together with the next *cap functional equation*:

$$\exists\, \widehat{g}(\cdot) \in \widehat{\mathcal{G}} :\ \exists \int_a^b f(t)\widehat{g}'(t)dt = 0. \tag{36}$$

We are finally ready to present our main result: a necessary and sufficient condition for the existence of a solution to (36), namely the existence of a solution to the next *weight equation*:

$$\exists\, w(\cdot) \in \mathcal{W}^f :\ \ 0 \in f_{ab}^w((a,b)). \tag{37}$$

We present first, in (38), our main theoretical result and later on, in (40), our main result for application to explicitly given functions $f(\cdot)$.

Theorem 1. *Given $f(\cdot)$ as in (10),*

$$(36) \qquad \textit{is equivalent to} \qquad (37). \tag{38}$$

Proof. (a) Assume first (37), namely that

$$\exists\, w(\cdot) \in \mathcal{W}^f \ \ \&\ \ \exists\, m \in (a,b) :\ f_a^w(m) = f_b^w(m);$$

and define the function $\widehat{g}(\cdot) \in W_0^{1,1}([a,b])$ by

$$\widehat{g}(t) := \begin{cases} \dfrac{1}{W_a(m)} \displaystyle\int_a^t w(\tau)d\tau & for\ t \in [a,m] \\ \dfrac{1}{W_b(m)} \displaystyle\int_t^b w(\tau)d\tau & for\ t \in [m,b], \end{cases}$$

obtaining:

$$\widehat{g}'(t) = \begin{cases} w(t)/W_a(m) & for\ a.e.\ t \in (a,m) \\ -w(t)/W_b(m) & for\ a.e.\ t \in (m,b). \end{cases}$$

Therefore

$$\begin{aligned} &\int_a^b f(t)\widehat{g}'(t)dt = \frac{1}{W_a(m)} \int_a^m f(t)w(t)dt - \frac{1}{W_b(m)} \int_m^b f(t)w(t)dt \\ &= \left[f_a^w(m) + \overline{f}^w\right] - \left[f_b^w(m) + \overline{f}^w\right] = (b-a)f_{ab}^w(m) = 0, \end{aligned}$$

by the hypothesis (37) assumed in this first part of the proof. We have thus proved that

$$(37) \qquad implies \qquad (36). \tag{39}$$

(b) Assuming now (36), we have

$$\exists\, \widehat{g}(\cdot) \in W_0^{1,1}([a,b]) :\ \left(f.\widehat{g}'\right)(\cdot) \in L^1(a,b)\ \ \&\ \ \widehat{g}(t) > 0\ \ \forall t \in (a,b)$$

$$\exists\, m \in (a,b) :\ \widehat{g}(m) = M\ \ \&\ \ (35)\ \ \&\ \int_a^b f(t)\widehat{g}'(t)dt = 0.$$

Defining the weight-function

$$w(\cdot) := \frac{1}{2M}\left|\widehat{g}'(\cdot)\right| \quad \text{we get} \quad w(t) = \begin{cases} \widehat{g}'(t)/2M & \text{a.e. in } (a,m) \\ -\widehat{g}'(t)/2M & \text{a.e. in } (m,b), \end{cases}$$

and clearly

$$t \in [a,m] \Rightarrow W_a(t) := \int_a^t w(\tau)d\tau = \frac{1}{2M}\int_a^t \widehat{g}'(\tau)d\tau = \widehat{g}(t)/2M$$
$$t \in [m,b] \Rightarrow W_b(t) = \int_t^b w(\tau)d\tau = -\frac{1}{2M}\int_b^t -\widehat{g}'(\tau)d\tau = \widehat{g}(t)/2M,$$

hence

$$W_a(t) = \begin{cases} \widehat{g}(t)/2M & \text{for } t \in [a,m] \\ 1 - W_b(t) = 1 - \widehat{g}(t)/2M & \text{for } t \in [m,b] \end{cases}$$

$$W_b(t) = \begin{cases} 1 - W_a(t) = 1 - \widehat{g}(t)/2M & \text{for } t \in [a,m] \\ \widehat{g}(t)/2M & \text{for } t \in [m,b] \end{cases}$$

$$W_a(m) = \frac{1}{2} = W_b(m) \quad \& \quad W_a(b) = 1 = W_b(a),$$

in particular $w(\cdot) \in \mathcal{W}^f$; and, finally,

$$f_{ab}^w(m) := \frac{f_b^w - f_a^w}{b-a}(m) = \frac{2}{b-a}\left[\int_m^b f(t)w(t)dt - \int_a^m f(t)w(t)dt\right]$$
$$= \frac{1}{b-a}\left[-\int_b^m -f(t)\widehat{g}'(t)/M\,dt - \int_a^m f(t)\widehat{g}'(t)/M\,dt\right]$$
$$= -\frac{1}{M(b-a)}\left[\int_a^m f(t)\widehat{g}'(t)dt + \int_m^b f(t)\widehat{g}'(t)dt\right] = 0,$$

by the hypothesis (36) assumed in this second part of the proof. We have thus proved the opposite implication of (39), hence (38), concluding the proof of Theorem 1. □

Theorem 2. *Given $f(\cdot)$ as in (10), for the $\overline{x}(\cdot)$ fixed in (4) we have:*

$$\begin{aligned} &(22), (37) \text{ and } w(\cdot) \in L^\infty(a,b) \Rightarrow \\ &\Rightarrow \overline{\mathcal{X}}_-^{bb} \neq \emptyset \Leftrightarrow f(\cdot) \text{ is good (see (27))}. \end{aligned} \tag{40}$$

Proof. Clearly, due to (25),

$$\begin{aligned} &(22) \;\&\; (37) \text{ with } w(\cdot) \in L^\infty(a,b) \Rightarrow \\ &\Rightarrow (36)\&(24)\&(23) \Rightarrow (17) \Leftrightarrow \overline{\mathcal{X}}_-^{bb} \neq \emptyset. \end{aligned}$$ □

Remark 1. *(Why is (38) a wonderful result?)*

The main reason is that either (36) or our previous algebraic conditions in [12] turn out very difficult to apply in practice, i.e., to explicitly given functions $f(\cdot)$; while, on the contrary, our new geometrical condition (37) is very very easy to apply, as the reader will see in Section 5. Indeed, there we present several explicit $f(\cdot)$ to which one very easily applies (37).

We return to this discussion in the final Section 6, after readers had the opportunity of gaining some intuitive feeling.

3. Some Properties of Lateral Weighted-Averages

We claim that for $w(\cdot) \in \mathcal{W}^f$:

$$f_a^w(\cdot) \in W^{1,1}_{loc}((a,b]) \quad \& \quad f_b^w(\cdot) \in W^{1,1}_{loc}([a,b)) \tag{41}$$

$$f_{ab}^w(\cdot) \in W^{1,1}_{loc}((a,b)) \tag{42}$$

$$f(\cdot) \in L^\infty(a,b) \ \Rightarrow \ |f_a^w(\cdot)|_\infty \le 2|f(\cdot)|_\infty \quad \& \quad |f_b^w(\cdot)|_\infty \le 2|f(\cdot)|_\infty. \tag{43}$$

Indeed, while (41) and (42) follow at once from their definitions, (43) follows from application of the obvious inequality

$$\left| f_a^w(t) + \overline{f}^w \right| = \frac{1}{W_a(t)} \left| \int_a^t f(\tau)w(\tau)d\tau \right| \le |f(\cdot)|_\infty \frac{W_a(t)}{W_a(t)} \quad \forall t \in (a,b];$$

and similarly for $\left| f_b^w(\cdot) \right|_\infty$. On the other hand,

$$\begin{pmatrix} f(\cdot),\ w(\cdot) \\ \text{continuous at } \ t=a \\ \text{with } \ w(a) \neq 0 \end{pmatrix} \Rightarrow \begin{pmatrix} f_a^w(\cdot) \in C^0([a,b]) \\ \& \ f_a^w(a) = f(a) - \overline{f}^w \end{pmatrix} \tag{44}$$

$$\begin{pmatrix} f(\cdot),\ w(\cdot) \\ \text{continuous at } \ t=b \\ \text{with } \ w(b) \neq 0 \end{pmatrix} \Rightarrow \begin{pmatrix} f_b^w(\cdot) \in C^0([a,b]) \\ \& \ f_b^w(b) = f(b) - \overline{f}^w \end{pmatrix}, \tag{45}$$

by application of L'Hôpital's rule (see [17] (Proposition 10.5.1) with [18] (Lemma 13.14)):

$$\lim_{t \searrow a} \left[f_a^w(t) + \overline{f}^w \right] = \lim_{t \searrow a} \frac{\int_a^t f(\tau)w(\tau)d\tau}{\int_a^t w(\tau)d\tau} = \lim_{t \searrow a} \frac{f(t)w(t)}{w(t)} = f(a),$$

and similarly for $f_b^w(\cdot)$. Notice that there are more general hypotheses, based on essential continuity, also implying the *rhs* of (44) and (45), see e.g., [19].

Therefore, in general—i.e., except when the *rhs* of (44) and (45) hold true—$f_b^w(t)$ starts from the value zero at $t = a$ and proceeds absolutely continuously along (a,b), but may oscillate wildly as $t \nearrow b$, to the point of having no definite value at $t = b$; while, in contrast, also $f_a^w(t)$ starts from the value zero, but assumed rather at the other endpoint $t = b$, and proceeds absolutely continuously along (a,b), backwards in time, possibly also oscillating wildly but as $t \searrow a$ instead, maybe to the point of possessing no definite value at the endpoint $t = a$.

This is why we prefer to define the value of $f_a^w(\cdot)$ at $t = a$, and the value of $f_b^w(\cdot)$ at $t = b$, not as single values but rather as oscillation intervals:

$$f_a^w(a) := \left[\liminf_{t \searrow a} f_a^w(t),\ \limsup_{t \searrow a} f_a^w(t) \right] \tag{46}$$

$$f_b^w(b) := \left[\liminf_{t \nearrow b} f_b^w(t),\ \limsup_{t \nearrow b} f_b^w(t) \right]. \tag{47}$$

As to $f_{ab}^w(\cdot)$: there's no need of considering its values at the endpoints a & b, since we will need to consider only its image $f_{ab}^w((a,b))$, recall (34), hence definition (33) suffices.

Assuming $f(\cdot) \in L^1(a,b)$, the simplest versions of weighted-averages (31) are obviously the plain averages computed using the constant weight, i.e., $w(\cdot) \equiv 1/(b-a)$:

$$\begin{aligned} & f_a(t) := \frac{1}{t-a}\int_a^t f(\tau)d\tau - \overline{f} \quad \& \quad f_b(t) := \frac{1}{b-t}\int_t^b f(\tau)d\tau - \overline{f} \\ & \& \quad f_{ab}(t) := \frac{f_b(t) - f_a(t)}{b-a}, \quad with \quad \overline{f} := \frac{1}{b-a}\int_a^b f(t)dt. \end{aligned} \tag{48}$$

A simple example of interval (46) with nonempty interior is the one associated to

$$f(t) := \sin\left(\frac{\pi}{\sqrt{t}}\right) - \frac{\pi}{2\sqrt{t}}\cos\left(\frac{\pi}{\sqrt{t}}\right) + 2\sigma t, \ \sigma \in \mathbb{R}, \tag{49}$$

which satisfies (41) but not (44), with $a = 0$ & $b = 1$:

$$f_a(t) = \begin{cases} \sin\left(\frac{\pi}{\sqrt{t}}\right) + (t-1).\sigma & for\ t \in (0,1] \\ [-\sigma-1, -\sigma+1] & at\ t = 0 \end{cases} \tag{50}$$

$$f_b(t) = \begin{cases} \sigma t - \frac{\sin\left(\frac{\pi}{\sqrt{t}}\right)}{1/t-1} & for\ t \in (0,1) \\ 0 & at\ t = 0 \\ \sigma + \frac{\pi}{2} & at\ t = 1 \end{cases} \tag{51}$$

$$f_{ab}(t) = \sigma - \frac{1}{1-t}\sin\left(\frac{\pi}{\sqrt{t}}\right) \ \rightarrow \ \sigma + \pi/2 \ \ as \ \ t \nearrow 1.$$

Notice that while $f(\cdot) \in L^1 \backslash L^\infty(a,b)$—namely does not satisfy the *lhs* of (43)—we still have $f_a(\cdot) \in L^\infty(a,b)$ & $f_b(\cdot) \in L^\infty(a,b)$ & $f_{ab}(\cdot) \in L^\infty(a,b)$.

Finally, we complete this Section 3 with some helpful rules to classify critical points of $f_a^w(\cdot)$ & $f_b^w(\cdot)$. To begin with, the set

$$\mathcal{N} := \left\{ \begin{array}{ll} t \in [a,b]: & \nexists\ \frac{d}{dt}\int_a^t f(\tau)w(\tau)d\tau \\ & or\ \ \frac{d}{dt}\int_a^t f(\tau)w(\tau)d\tau \neq f(t)w(t) \\ & or\ \nexists\ W_a'(t)\ \ or\ \ W_a'(t) \neq w(t) \end{array} \right\}$$

has zero measure and:

$$\frac{d}{dt} f_a^w(t) = w(t)\frac{f(t) - \overline{f}^w - f_a^w(t)}{W_a(t)} \ \ \forall t \in (a,b] \backslash \mathcal{N} \tag{52}$$

$$\frac{d}{dt} f_b^w(t) = -w(t)\frac{f(t) - \overline{f}^w - f_b^w(t)}{W_b(t)} \ \ \forall t \in [a,b) \backslash \mathcal{N}. \tag{53}$$

Hence, with $\mathcal{M} := \{t \in [a,b] \backslash \mathcal{N} : \ \exists\, w'(t) \in \mathbb{R} \ \ \& \ \ \exists\, f'(t) \in \mathbb{R}\}$ and

$$\begin{aligned} C_a &:= \left\{ t \in (a,b] \backslash \mathcal{N} : \ f_a^w(t) = f(t) - \overline{f}^w \right\} \\ C_b &:= \left\{ t \in [a,b) \backslash \mathcal{N} : \ f_b^w(t) = f(t) - \overline{f}^w \right\}, \end{aligned}$$

we have

$$\exists\ \frac{d^2}{dt^2} f_a^w(t) = w(t)\frac{f'(t)}{W_a(t)} \ \ \forall t \in \mathcal{M} \cap C_a$$

$$\exists \; \frac{d^2}{dt^2} f_b^w(t) = -\,w(t) \frac{f'(t)}{W_b(t)} \;\; \forall t \in \mathcal{M} \cap C_b$$

$$\begin{aligned} f_a^w(\cdot) \quad & \textit{increases where} \;\; f_a^w(\cdot) \le f(\cdot) - \overline{f}^{\,w} \;\; \& \\ & \textit{decreases where} \;\; f_a^w(\cdot) \ge f(\cdot) - \overline{f}^{\,w} \end{aligned}$$

$$\begin{aligned} f_b^w(\cdot) \quad & \textit{decreases where} \;\; f_b^w(\cdot) \le f(\cdot) - \overline{f}^{\,w} \;\; \& \\ & \textit{increases where} \;\; f_b^w(\cdot) \ge f(\cdot) - \overline{f}^{\,w}. \end{aligned}$$

Here, increases (respectively decreases) is meant along each open interval.

Clearly

$$\begin{aligned} & \textit{the critical points of} \;\; f_a^w(\cdot) \;\; \textit{in} \; (a,b] \backslash \mathcal{N} \\ & \textit{are those} \;\; c \in \{t \in (a,b] \backslash \mathcal{N} : \; w(t) = 0\} \cup C_a \end{aligned} \tag{54}$$

$$\begin{aligned} & \textit{the critical points of} \;\; f_b^w(\cdot) \;\; \textit{in} \; [a,b) \backslash \mathcal{N} \\ & \textit{are those} \;\; c \in \{t \in [a,b) \backslash \mathcal{N} : \; w(t) = 0\} \cup C_b; \end{aligned} \tag{55}$$

while for those critical points c in (54) and (55) where $w(c) > 0$ we have

$$\begin{aligned} f'(c) < 0 \implies c \;\; \textit{is a local} \;\; \max \;\; \textit{point of} \;\; f_a^w(\cdot) \\ f'(c) > 0 \implies c \;\; \textit{is a local} \;\; \min \;\; \textit{point of} \;\; f_a^w(\cdot) \end{aligned}$$

$$\begin{aligned} f'(c) < 0 \implies c \;\; \textit{is a local} \;\; \min \;\; \textit{point of} \;\; f_b^w(\cdot) \\ f'(c) > 0 \implies c \;\; \textit{is a local} \;\; \max \;\; \textit{point of} \;\; f_b^w(\cdot). \end{aligned}$$

4. Alternative Sufficient Conditions for (37)

To guarantee the existence of bang-bang solutions in specific examples of our pointwise-constrained linear control system with $w(\cdot) \in L^\infty(a,b)$, assuming (22) and using explicitly given functions $f(\cdot)$ in (10), one easily applies (37), in most cases.

However, in some of the examples presented in the next section, we have felt to be even simpler the application of one of the next sufficient—but not necessary—conditions for (37).

Our first such condition is:

$$\exists\, w(\cdot) \in \mathcal{W}^f \;\; \exists\, c, d \in (a,b) : \;\; f_a^w(c).\, f_b^w(d) > 0, \tag{56}$$

which is satisfied, in particular, provided

$$\begin{aligned} & \exists\, w(\cdot) \in \mathcal{W}^f : \;\; f_a^w(a).\, f_b^w(b) > 0 \;\; \textit{or}, \\ & \textit{more precisely, due to (46) \& (47)}, \\ & \exists\, v_a \in f_a^w(a) \;\; \& \;\; \exists\, v_b \in f_b^w(b) : \;\; v_a \,.\, v_b > 0; \end{aligned} \tag{57}$$

a special case of which, applicable whenever the *lhs* of (44) and (45) both hold true, is

$$\exists\, w(\cdot) \in \mathcal{W}^f : \;\; \left(f(a) - \overline{f}^{\,w} \right) . \left(f(b) - \overline{f}^{\,w} \right) > 0. \tag{58}$$

And here is our second such sufficient condition:

$$\exists\, w(\cdot) \in \mathcal{W}^f : \;\; 0 \in f_a^w((a,b)) \;\; \textit{or} \;\; 0 \in f_b^w((a,b)). \tag{59}$$

Finally, we present some negative conditions, i.e., conditions that—instead of implying (37), as the preceding ones—imply the negation of (37):

$$f((a,b)) \;\; \textit{has positive measure} \tag{60}$$

to be used together with one of the following:

$$f(\cdot) \ \ is\ monotonic\ along \ \ (a,b) \tag{61}$$

$$\begin{aligned} f_a^w(\cdot) \ \ \& \ \ & f_b^w(\cdot) \ \ both\ increase\ along \ \ (a,b) \ \ \forall w(\cdot) \in \mathcal{W}^f \\ & \& \ \ f(E) \ \ is\ a\ null\ set\ for\ each\ null\ set \ \ E \subset (a,b) \end{aligned} \tag{62}$$

$$\begin{aligned} f_a^w(\cdot) \ \ \& \ \ & f_b^w(\cdot) \ \ both\ decrease\ along \ \ (a,b) \ \ \forall w(\cdot) \in \mathcal{W}^f \\ & \& \ \ f(E) \ \ is\ a\ null\ set\ for\ each\ null\ set \ \ E \subset (a,b). \end{aligned} \tag{63}$$

Theorem 3. *Under* (30),

$$(56) \ \ (or \ \ (57) \ \ or \ \ (44) \ \& \ (45) \ \& \ (58) \ \ or \ \ (59)) \ \ \Rightarrow \ \ (37) \tag{64}$$

$$(60) \ \& \ (61) \ \ \Rightarrow \ \ negation\ of\ (37) \tag{65}$$

$$\left\{ \begin{array}{l} (60)\ and \\ (either\ (62)\ or\ else\ (63)) \end{array} \right\} \ \Rightarrow \ \ negation\ of\ (37). \tag{66}$$

Proof. (a) In order to prove (64): assuming (56) certainly $c \neq d$, by (34). If $f_a^w(c) > 0$ & $f_b^w(d) > 0$ then, again by (34), $f_b^w(c) \neq f_a^w(c) > 0$ & $f_a^w(d) \neq f_b^w(d) > 0$, which implies that $f_b^w(c) < 0$ & $f_a^w(d) < 0$. Therefore

$$f_a^w(c) > 0 \ \ \& \ \ f_a^w(d) < 0 \ \ \& \ \ f_b^w(c) < 0 \ \ \& \ \ f_b^w(d) > 0.$$

By continuity and (34):

$$\exists \bar{t} \in co\{c,d\}: \ \ f_a^w(\bar{t}) = 0 \ \ \& \ \ f_b^w(\bar{t}) = 0 \ \Rightarrow \ (37).$$

The case $f_a^w(c) < 0$ & $f_b^w(d) < 0$ is similar.

Considering now (57), necessarily

$$\exists \ \ convergent\ sequences \ \ (a_k) \searrow a, \ (b_k) \nearrow b$$

with

$$\begin{aligned} & \left(v_a^k := f_a^w(a_k)\right) \to v_a \ \ \& \ \ \left(v_b^k := f_b^w(b_k)\right) \to v_b \\ & \quad \& \ either \ \ \left\{v_a^k > 0 \ \& \ v_b^k > 0 \ \ \forall k\right\} \\ & \quad or\ else \ \ \ \ \left\{v_a^k < 0 \ \& \ v_b^k < 0 \ \ \forall k\right\}. \end{aligned} \tag{67}$$

Assuming either alternative in (67), one easily checks, using the reasoning above, that the graphs of $f_a^w(\cdot)$ and of $f_b^w(\cdot)$ must intersect on the straight-line segment $(a,b) \times \{0\}$, hence (37).

In order to prove that also (59) implies (37), assume e.g., $0 \in f_b^w((a,b))$. Then

$$\exists \bar{t} \in (a,b): \ \ f_b^w(\bar{t}) = 0$$

which implies, by (34), that $f_a^w(\bar{t}) = 0$, hence (37). This proves (64).

(b) To prove (65), assume (60) and $f(\cdot)$ increasing along (a,b), otherwise just replace $f(\cdot)$ by $-f(\cdot)$, and take any $w(\cdot) \in \mathcal{W}^f$. Then, the next inequalities are obvious also from the geometrical viewpoint:

$$\frac{\displaystyle\int_a^t f(\tau)w(\tau)d\tau}{\displaystyle\int_a^t w(\tau)d\tau} \leq f(t) \leq \frac{\displaystyle\int_t^b f(\tau)w(\tau)d\tau}{\displaystyle\int_t^b w(\tau)d\tau} \ \ \ \forall t \in (a,b). \tag{68}$$

They just tell us that $f(\cdot)$—being increasing—always moves faster (respectively slower) than past (respectively future) averages: indeed, the average computed with t moving up from $t = a$ (respectively moving down from $t = b$), is always below (respectively above) the value $f(t)$.

But these two inequalities imply—recall the expressions (52) and (53) for the derivatives of the lateral averages—that

$$\frac{d}{dt} f_a^w(t) = w(t)\frac{f(t) - \overline{f}^w - f_a^w(t)}{W_a(t)} \geq 0 \ \ a.e. \quad \& \\ \frac{d}{dt} f_b^w(t) = -w(t)\frac{f(t) - \overline{f}^w - f_b^w(t)}{W_b(t)} \geq 0 \ \ a.e.,$$

as one easily checks, since $w(t) \geq 0$ *a.e.* & $W_a((a,b)) = (0,1) = W_b((a,b))$.

To prove the $\Rightarrow$ in (65), one just has to realize that (60), (68) and (34) imply

$$f_a^w(t) < 0 \ \ \forall t \in (a,b) \quad \textit{since it increases to} \quad f_a^w(b) = 0, \\ f_b^w(t) > 0 \ \ \forall t \in (a,b) \quad \textit{since it increases from} \quad f_b^w(a) = 0.$$

But this means that

$$f_{ab}^w(t) := \frac{f_b^w(t) - f_a^w(t)}{b-a} > 0 \quad \forall t \in (a,b),$$

thus denying (37). The proof of (66) is similar. □

5. Examples of Application of Theorems 2 and 3 to Check Goodness of Some Explicitly Given Functions $f(\cdot)$

Theorem 4. *Consider any function $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\exists\, t_0 \in (a,b) : \ \ f(t) = f(t_0) + \sum_{j=1}^{\infty} d_j.(t-t_0)^{2j} \quad \forall t \in [a,b] \tag{69}$$

with $d_j \geq 0\ \forall j \in \mathbb{N}$ & $\exists\, j_0 \in \mathbb{N}$ such that $d_{j_0} > 0$.

Then (37) holds true. Moreover, under "good data", recall (26), $f(\cdot)$ is good, see (27), whenever

$$\max\left\{ \frac{f(a) - f(t_0)}{2(f(b) - f(t_0))}, \frac{f(b) - f(t_0)}{2(f(a) - f(t_0))} \right\} < 1. \tag{70}$$

Notice that if, in (69), $d_j = 0\ \forall j \in \mathbb{N}$ then clearly $f(\cdot)$ is constant hence is good, as stated in (28).

Remark 2. *Under the hypotheses of Theorem 4, the inequality (70) holds true in particular whenever $f(a) = f(b)$, or (equivalently) $t_0 = (a+b)/2$. In such case, it is easy to determine $\widehat{g}(\cdot)$ satisfying (36); hence, under "good data", $\widetilde{g}(\cdot)$ satisfying (18).*

Corollary 1. *Consider any function $f : \mathbb{R} \to \mathbb{R}$ that is represented by a convergent power series on an interval of positive radius centered at some t_0, i.e.,*

$$\exists\, t_0 \in \mathbb{R} \quad \exists\, \eta \in (0,\infty] : \ \ f(t) = f(t_0) + \sum_{j=1}^{\infty} c_j.(t-t_0)^j \quad \forall t \in (t_0 - \eta, t_0 + \eta),$$

satisfying the hypotheses:

$$f(t_0 - r) = f(t_0 + r) \ \ \forall r \in (0,\delta) \quad \textit{for some} \quad 0 < \delta \leq \eta \\ f^{(j)}(t_0) \geq 0 \ \ \forall j \in \mathbb{N} \quad \& \quad \exists\, j_0 \in \mathbb{N} : \ f^{(j_0)}(t_0) > 0.$$

If $t_0 - \eta < a < t_0 < b < t_0 + \eta$ then the conclusions of Theorem 4 hold true.

Proof of Theorem 4. Clearly

$$f_0(t) := f(t) - f(t_0) = \sum_{j=1}^{\infty} d_j.(t - t_0)^{2j} \quad \forall t \in [a, b],$$

with $t_0 \in (a, b)$, $d_j \geq 0\ \forall j$ & $d_{j_0} > 0$; hence one obtains, with

$$v_k(t) := |t - t_0|^{-1+\frac{1}{k}},$$

$$|v_k(\cdot)|_1 := \int_a^b |t - t_0|^{-1+\frac{1}{k}} dt = \int_a^{t_0} (t_0 - t)^{-1+\frac{1}{k}} dt + \int_{t_0}^b (t - t_0)^{-1+\frac{1}{k}} dt$$
$$= k.\left[(t_0 - a)^{1/k} + (b - t_0)^{1/k}\right] \geq k.(b - a)^{1/k} > 0$$

since, with $\alpha := t_0 - a > 0$ & $\beta := b - t_0 > 0$, hence $\alpha + \beta = b - a$, for $k \geq 2$

$$\left(\alpha^{1/k} + \beta^{1/k}\right)^k = \alpha + \beta + \sum_{i=1}^{k-1} \frac{k!}{i!\,(k-i)!}\, \alpha^{i/k} \beta^{\frac{k-i}{k}} > \alpha + \beta.$$

Therefore

$$\frac{1}{2} \cdot \frac{1}{k} \cdot \frac{1}{(b-a)^{1/k}} \leq \frac{1}{|v_k(\cdot)|_1} \leq \frac{1}{k} \cdot \frac{1}{(b-a)^{1/k}}.$$

Setting now

$$w_k(t) := \frac{v_k(t)}{|v_k(\cdot)|_1}, \quad \textit{yielding} \quad \int_a^b w_k(t) dt = 1,\ \forall k \in \mathbb{N},$$

one gets

$$\frac{1}{2} \cdot \frac{1}{k} \cdot \frac{|t - t_0|^{-1+\frac{1}{k}}}{(b-a)^{1/k}} \leq w_k(t) \leq \frac{1}{k} \cdot \frac{|t - t_0|^{-1+\frac{1}{k}}}{(b-a)^{1/k}},$$

$$\int_a^{t_0} f_0(t) w_k(t) dt = \frac{1}{|v_k(\cdot)|_1} \sum_{j=1}^{\infty} d_j . \int_a^{t_0} (t_0 - t)^{2j-1+1/k} dt$$
$$= \frac{(t_0 - a)^{1/k}}{|v_k(\cdot)|_1} \sum_{j=1}^{\infty} d_j \frac{(t_0 - a)^{2j}}{2j + 1/k} \leq \frac{1}{k} \sum_{j=1}^{\infty} d_j \frac{(a - t_0)^{2j}}{2 + 1/k},$$

hence

$$\int_a^{t_0} f_0(t) w_k(t) dt \leq \frac{f_0(a)}{2k+1}.$$

On the other hand since, similarly,

$$\int_{t_0}^b f_0(t) w_k(t) dt \leq \frac{f_0(b)}{2k+1},$$

one finally gets

$$\int_a^b f_0(t) w_k(t) dt \leq \frac{f_0(a) + f_0(b)}{2k+1}$$

and, setting

$$w(\cdot) := w_{k_0}(\cdot), \quad \textit{where} \quad k_0 := \textit{any integer} > \max\left\{\frac{f_0(a)}{2f_0(b)}, \frac{f_0(b)}{2f_0(a)}\right\}, \tag{71}$$

one obtains

$$\overline{f}_0^{\,w} := \int_a^b f_0(t) w(t) dt < \min\{f_0(a), f_0(b)\};$$

and since the same inequality also holds for $f(\cdot)$ itself—as is obvious—one finally reaches the inequality

$$\Big(f(a)-\overline{f}^{w}\Big).\Big(f(b)-\overline{f}^{w}\Big)>0.$$

Therefore, one may apply (64) to conclude that (37) holds true. Then, assuming (70), in (71) we can take $k_0 := 1$ yielding $w(\cdot) \equiv 1/(b-a)$ hence, under (22), $f(\cdot)$ is good, as in (40). □

Example 1. *(How much non-monotonic must a piece of parabola be so as to be good using the constant weight?)*

Consider in (10) the simplest non-monotonic $f(\cdot) \in C^0(\mathbb{R})$, namely $f(t) = 3t^2$; together with the simplest of the weighted-averages in (31), namely the weighted-averages (48) using the constant weight. In order for $f|_{(a,b)}(\cdot)$ to be good—in the sense (27)—the most obvious thought that comes to one's mind is, by (29) (see also (65)), that "$f(\cdot)$ cannot be strictly monotonic", which forces one to pick

$$a<0<b \quad hence\ start\ by\ fixing\ some \quad a\in(-\infty,0)$$

and then look for those

$$b=b(a)\in(0,\infty) \quad which\ are\ good,\ as\ above.$$

Using (70) one obtains

$$b\in\Big(|a|/\sqrt{2},\sqrt{2}\,|a|\Big) \ \Rightarrow\ 0\in f_{ab}((a,b)).$$

However, (70) is sufficient but not necessary for (37) and, in fact, (48) yields

$$\begin{array}{l} f_a(t)=(t-b)f_{ab}(t) \quad \& \quad f_b(t)=(t-a)f_{ab}(t) \\ with \quad f_{ab}(t)=t+a+b \quad \& \quad \overline{f}=a^2+ab+b^2. \end{array}$$

Then the condition (37) in the rhs of the equivalence (38) becomes, using the constant weight hence (48),

$$0\in f_{ab}((a,b)) \ \Leftrightarrow\ b\in(|a|/2\,,2|a|) \tag{72}$$

which, assuming "good data" as in (26), is a sufficient condition for goodness of $f(\cdot)$, as in (40).

Notice, by the way, that application of the condition (57), or (58), with $w(\cdot)$ constant in the lhs of (64) is possible if and only if

$$\begin{array}{rcl} f_a(a).\,f_b(b) & = & 2(b-a)^2(2|a|-b).(b-|a|/2)>0 \\ & \Leftrightarrow & |a|/2<b<2|a|. \end{array}$$

Since the conclusion (72) has been reached using only the constant weight, the next question that comes to one's mind is: could this result be improved by using more sophisticated weights so as to arbitrarily enlarge the interval in the *rhs* of (72)? Or, in other words, could any non-monotonic parabola-arc be considered good using an adequate weight function? The answer is yes, as follows from the more general

Theorem 5. *($f(\cdot)$ non-monotonic may be good)*

Let $f(\cdot) \in L^\infty(a,b)$ satisfy:

$$\begin{array}{l} \exists\, m<M\ in\ \mathbb{R} \quad \exists\, \varepsilon>0 \quad \exists\, E\subset[a+\varepsilon,b-\varepsilon]\ \ with\ \ |E|>0: \\ f(t)\le m\ \ for\ a.e.\ t\in E \quad \& \\ M\le f(t)\ for\ a.e.\ t\in(a,a+\varepsilon)\cup(b-\varepsilon,b). \end{array} \tag{73}$$

Then

$$\text{under ``good data''}, \quad f(\cdot) \ \text{ is good} \quad (\text{recall } (26) \ \& \ (27)). \tag{74}$$

Remark 3. *Roughly speaking, (73) asks that the "lowest" values of $f(\cdot) \in L^\infty(a,b)$ should be assumed a.e. away from the boundary $\{a,b\}$. But "lowest" can here be replaced by "highest", since changing $f(\cdot)$ to $-f(\cdot)$ in (73) does not affect*

$$\text{neither} \quad \int_a^b f(t)g'(t)dt = 0 \qquad \text{nor} \qquad 0 \in f_{ab}^w((a,b)).$$

Again roughly speaking, this means that $f(\cdot)$ being "strictly monotonic" is the worst possible behaviour. Any $f(\cdot) \in C^0([a,b])$ will satisfy (73) provided

$$\begin{array}{ll} \text{either} & \{t \in [a,b] : \ f(t) = \min f([a,b])\} \subset (a,b) \\ \text{or} & \{t \in [a,b] : \ f(t) = \max f([a,b])\} \subset (a,b). \end{array} \tag{75}$$

Proof of Theorem 5. Let us find an adequate $w(\cdot) \in \mathcal{W}^f \cap L^\infty(a,b)$ for which $0 \in f_{ab}^w((a,b))$ hence $f(\cdot)$ is good, since we are assuming (22). We can choose $w(\cdot) > 0$ a.e. so small in $[a,b]\setminus E$ so as to get $\delta := \int_{[a,b]\setminus E} w(t)dt$ as small as desired. Then

$$\begin{aligned} \overline{f}^w &:= \int_a^b f(t)w(t)dt = \int_{[a,b]\setminus E} f(t)w(t)dt + \int_E f(t)w(t)dt \\ &\leq |f(\cdot)|_\infty.\delta + m.(1-\delta) = m + \delta.(|f(\cdot)|_\infty - m) < M, \end{aligned}$$

proving that

$$f_a^w(a+\varepsilon) = \frac{\displaystyle\int_a^{a+\varepsilon} f(t)w(t)dt}{\displaystyle\int_a^{a+\varepsilon} w(t)dt} - \overline{f}^w \geq M - \overline{f}^w > 0$$
$$f_b^w(b-\varepsilon) = \frac{\displaystyle\int_{b-\varepsilon}^{b} f(t)w(t)dt}{\displaystyle\int_{b-\varepsilon}^{b} w(t)dt} - \overline{f}^w \geq M - \overline{f}^w > 0.$$

Therefore $f_a^w(a+\varepsilon).f_b^w(b-\varepsilon) > 0$ hence, by (64), $0 \in f_{ab}^w((a,b))$ as desired, thus (by (40)) completing the proof. □

Example 2. *(General formulas for analytic functions)*

Whenever the given $f(\cdot)$ is either a polynomial or analytic function, namely of the form

$$f^N(t) := \sum_{k=0}^{N} (k+1)c_k.t^k, \quad \text{with} \ \ N \in \mathbb{N} \cup \{\infty\},$$

then—as one may easily check—its weighted-averages (48) can be written as

$$\begin{aligned} & f_a^N(t) = (t-b)f_{ab}^N(t) \quad \& \quad f_b^N(t) = (t-a)f_{ab}^N(t) \\ & \text{with} \quad f_{ab}^N(t) = \sum_{k=1}^{N} c_k \sum_{i=0}^{k-1} \frac{b^{k-i} - a^{k-i}}{b-a} t^i, \end{aligned} \tag{76}$$

a polynomial of degree $N-1$ (respectively a analytic function). Therefore the equation

$$0 \in f_{ab}^N((a,b))$$

has at most $N-1$ (respectively finitely many) solutions.

For example, $N = 2$ yields

$$f^2_{ab}(t) = c_2(a + b + t) + c_1 = 0 \Leftrightarrow t = -\frac{c_1}{c_2} - (a + b).$$

Therefore, assuming $a < 0$,

$$0 \in f^2_{ab}((a, b)) \Leftrightarrow b \in \left(\frac{|a|}{2} - \frac{c_1}{2c_2}, 2|a| - \frac{c_1}{c_2}\right)$$

(compare with (72), its special case $c_1 = 0$ & $c_2 = 1$); and, as could be expected, increasing N complicates expressions. With $N = 3$ one already needs to solve a quadratic equation:

$$f^3_{ab}(t) = c_3.t^2 + t[c_3(a + b) + c_2] + c_1 + c_2(a + b) + c_3\left(a^2 + ab + b^2\right) = 0.$$

Setting $c_3 := 1 =: -c_1$ & $c_2 := 0$, one gets

$$t = -\frac{a + b}{2} \pm \frac{1}{2}\sqrt{4 - (3a^2 + 3b^2 + 2ab)}$$

which only has a solution in case $3a^2 + 2ab + 3b^2 \leq 4$, namely for

$$b \in \left[\frac{-a}{3} - \frac{2}{3}\sqrt{3 - 2a^2}, \frac{-a}{3} + \frac{2}{3}\sqrt{3 - 2a^2}\right].$$

In particular, if $3a^2 + 2ab + 3b^2 = 4$,

$$0 \in f^3_{ab}((a, b)) \Leftrightarrow a \in \left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) \ \& \ b = \frac{-a}{3} + \frac{2}{3}\sqrt{3 - 2a^2}.$$

If, instead of applying directly (37), one wishes to apply (64), then (76) yields

$$f^N_a(a).\, f^N_b(b) = -(b - a)^2 f^N_{ab}(a)\, f^N_{ab}(b) > 0,$$

i.e., one needs to find conditions on $b = b(a)$ implying that

$$f^N_{ab}(a).\, f^N_{ab}(b) < 0.$$

Example 3. *(A wildly oscillating function $f(\cdot) \notin L^\infty(a, b)$ yielding a non-AC averaged-function $f_a(\cdot)$.)*

Let us go back to the example (49), in which the average $f_a(a)$ is not a point but rather a non-trivial interval, recall (50). This case presents a good test to check the applicability of our methods. Indeed, nothing is easy in this example but feasible, as we shall now see. In trying to figure out the easiest way to avoid complex computations, the first thing that comes to one's mind is to apply the test (57) with constant weight, since it involves fewer computations.

With $f_a(a)$ & $f_b(b)$ given as in (50) and (51), to fulfill (57) there are two possibilities: either $f_a(a) < 0$ ($\Leftrightarrow \sigma > -1$) & $f_b(b) < 0$ ($\Leftrightarrow \sigma < -\pi/2$), impossible; or else $f_a(a) > 0$ ($\Leftrightarrow \sigma < 1$) & $f_b(b) > 0$ ($\Leftrightarrow \sigma > -\pi/2$), i.e., $\sigma \in (-\pi/2, 1)$. This means that using the condition (57)—which is sufficient but not necessary for (37)—what one succeeds in finding is that

$$\sigma \in (-\pi/2, 1) \Rightarrow 0 \in f_{ab}((a, b)). \tag{77}$$

So, let us now try to apply directly condition (37), instead of condition (57), to check whether we can reach a sharper result. To begin with, it turns out convenient to define a new function

$$\psi : (0, 1) \to \mathbb{R}, \ \psi(t) := \frac{1}{1 - t} \sin\frac{\pi}{\sqrt{t}},$$

allowing us to write, more simply,

$$0 \in f_{ab}((a,b)) \;\Leftrightarrow\; \sigma \in \psi((a,b)).$$

It seems we are luckier with (37) than with (57) because, in fact, we now get a wider range-interval than the one in the lhs of (77).

Indeed, numerically—using e.g., the simple app MathStudio on a smartphone—we have found that

$$\psi((a,b)) = \psi((0,1)) = \psi(I) = J, \text{ where}$$
$$I := [0.16203\ldots, 0.58217\ldots] \;\&\; J := [-1.98206\ldots, 1.19190\ldots],$$

namely J strictly contains $(-\pi/2, 1)$ in (77).

This means that we do not need to look for zeros of $f_{ab}(\cdot)$ along the whole interval $(a,b) = (0,1)$, sufficing to search them in the subinterval I, which yields the same range of values for $\psi(\cdot)$. So, defining the inverse function $\alpha(\cdot)$ of the function $\psi|_I(\cdot)$,

$$\alpha : J \to I, \;\; t = \alpha(\sigma) \;\Leftrightarrow\; \sigma = \psi(t),$$

noticing that $\psi'(\cdot) < 0$ a.e. in $I \Leftrightarrow \alpha'(\cdot) < 0$ a.e. in J, we then have

$$t = \alpha(\sigma) \;\Leftrightarrow\; \sigma = \psi(t) \;\Rightarrow\; f_{ab}(t) = 0.$$

Hence

$$f_{ab}(\alpha(\sigma)) = 0 \;\; \forall \sigma \in J.$$

Thus, in conclusion, assuming "good data", as in (26), for values $\sigma \in J$ surely there exists a bang-bang solution for our pointwise-constrained linear first-order control system.

6. Discussion and Conclusions

Remark 4. *(Why is (37) much better than (36)?)*

Continuing the discussion started in Remark 1 above, notice that (36) is a functional equation, i.e., an equation whose unknown $\widehat{g}(\cdot)$ is a function, hence its difficulty is that we have no clue as to whether such function $\widehat{g}(\cdot)$ does—or does not—exist, and as to how could it be, if it existed. It may not be clear why we claim that (37) is a much better, geometrical, condition, since also (37) is a functional equation, its unknown being a weight-function $w(\cdot)$. So one might be tempted to ask: why would it be easier to find out whether such weight $w(\cdot)$ exists than whether such $\widehat{g}(\cdot)$ exists?

However, practical experience in trying to solve (37) in many specific examples of explicitly given functions $f(\cdot)$ shows that (37) is much easier than (36). Indeed, in many cases one can solve (37) without even worrying at all about looking for an adequate weight-function $w(\cdot)$: one just begins by applying the plain, obvious, constant weight $w(\cdot) \equiv 1/(b-a)$; and only in case this first, easy, attempt fails is one led to apply geometric intuition in order to try and find out what is wrong with this constant weight and what has to be done to improve it. A nice illustration of this appears in Example 1 above, dealing with non-monotonic parabola-arcs. Indeed, starting with the constant weight, one immediately finds the sufficient condition (72) for goodness under constant weight and "good data". And, after having developed our geometric intuition, we easily extended the same reasoning to a much wider class of functions, e.g., those $f(\cdot) \in C^0([a,b])$ whose min (or max) is not attained on the boundary $\{a,b\}$, recall (75) and (74).

As a bonus, this clearly shows why being $f(\cdot)$ strictly monotonic is the worst possible situation; and why any non-monotonic parabola-arc always satisfies (74). Would it be possible to reach such a general result directly from the theoretical result (36) instead of reaching it from its geometric version (37)? We doubt it.

As another example, who would guess, just by looking at (36)—not at (37)—the criteria presented in Example 2 above, for determining the goodness of general analytic functions relative to the values of their parameters? Finally, in Example 3 above—of a wildly oscillating function $f(\cdot) \in L^1 \backslash L^\infty(a,b)$, for which the corresponding weighted-average $f_a^w(\cdot)$ is not even AC on $[a,b]$, *depending on a parameter—who would guess good values for the parameter just by looking at the functional Equation* (36)*?*

Remark 5. *(Why is* (37) *much better than the algebraic conditions that we have presented in our previous paper [12]?)*

In [12] ((2.30), (2.31) and (3.52)) we have presented some algebraic conditions to ensure the existence of polynomial solutions $g(\cdot)$ *to the above functional Equation* (24)*. More precisely, solutions of the form* $g(t) = \varphi(t).\,\varphi(t).\,(t-a).(b-t)$*, for some polynomial* $\varphi(\cdot) \not\equiv 0$*. Our aim there was theoretical: to ensure that* (24) *has solutions, at least in some nontrivial situations. (Because otherwise* (24) *would be empty of meaning.) However, these algebraic conditions tend to become quite heavy in practice, i.e., when one wishes to apply them to specific examples of explicitly given functions* $f(\cdot)$ *as in* (10)*.*

Indeed, to apply them, one proceeds as follows. First, compute some main-diagonal entries M_f^{ii} *of a symmetric constant* $h \times h$ *-matrix* M_f *generated from* $f(\cdot)$ *through iterated indefinite integration, evaluated at the final endpoint* $t = b$*. To compute* M_f^{ii} *one needs to integrate* $2(i+1)$ *times. If one finds some* $i \in \{1, \ldots, h\}$ *for which* $M_f^{ii} = 0$ *or* $M_f^{ii}\,.\,M_f^{00} \leq 0$ *or* $M_f^{ii}.\,M_f^{00} \leq \left(M_f^{0i}\right)^2$*, then those algebraic conditions are satisfied. Otherwise, one may still compute the determinant of the* $h \times h$*-matrix and check whether it is* $= 0$*. Another possibility—in case the previous attempts fail—is to compute the eigenvalues of the* $h \times h$*-matrix to check whether two of them have a product* < 0*. In short, this is heavy work; and if one does not find a positive answer up to some* $i \in \{1, \ldots, h\}$*, for* $h = 1, 2, \ldots$*, then one never knows whether one step further—namely replace* h *by* $h+1$*—would deliver a positive answer. We have thus made our point clearly that those algebraic conditions are not very good from the pragmatic viewpoint of applying them in practice to a specific explicitly given* $f(\cdot)$*.*

In contrast, (37) *is incredibly better from this pragmatic viewpoint, as explained in Remark 4 above. Indeed, we do not see how could one possibly reach the fine details found in Examples 1–3 using only such algebraic conditions of [12], instead of* (37)*.*

In conclusion, the results of the present paper are much more powerful than previous results; hence we feel very proud of having invented/discovered them, since they certainly paid generously—in satisfaction—the hard work that we have invested into their invention/discovery.

Our most powerful result to ensure the goodness of an explicitly given $f(\cdot) \in L^\infty(a,b)$ under "good data" is undoubtedly Theorem 5, since to apply it one just needs to look at the graph of $f(\cdot)$, drawn either by hand or in a computer, tablet or smartphone. Indeed, if $f(\cdot) \in C^0([a,b])$ then one only has to check that minimal (respectively maximal) values are not assumed on the boundary; while for more general $f(\cdot) \in L^\infty(a,b)$ one should apply (73) and (74), either to $f(\cdot)$ or else to $-f(\cdot)$.

Author Contributions: Conceptualization, C.C., M.L. and A.O.; Formal analysis, C.C., M.L. and A.O.; Writing—review and editing, C.C., M.L. and A.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by FCT—Fundação para a Ciência e a Tecnologia in the framework of the research project "UIDB/04674/2020", DOI 10.54499/UIDB/04674/2020—CIMA (Centro de Investigação em Matemática e Aplicações).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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